Two tests of univariate stochastic dominance

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- Univariate stochastic dominance conditions depend on comparisons of statistics between two samples, A and B
- These statistics are linear combinations of cumulative probabilities
- There are several ways to test for stochastic dominance conditions. We will see two here: the KS test of Barrett and Donald (2003) and the test of Davidson and Duclos (2000)

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Davidson and Duclos (2000) found a very useful representation of $\Im_j(z; G)$:

$$\Im_j(z; G) = \frac{1}{(j-1)!} \int_0^z (z-y)^{j-1} dF(y)$$

Introduction continued

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$$\Im_j(z;\widehat{G}) = \frac{1}{N(j-1)!}\sum_{i=1}^{N}(z-y_i)^{j-1}I(y_i \leq z)$$

The test of Barrett and Donald (2003)

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- This test is a type of Kolmogorov-Smirnov test of homogeneity of distributions
- The logic is very intuitive: dominance conditions are based on comparing $\Im_j^A(z;G)$ with $\Im_j^B(z;G)$ for a range of z, where j is the order

The test of Barrett and Donald (2003)

There are four possible outcomes:

1. A dominates B: $\Im_j(z; G^A) \leq \Im_j^B(z; G^B) \forall z \in [0, z_{max}] \wedge \exists z | \Im_j(z; G^A) < \Im_j(z; G^B)$

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2. B dominates A: $\Im_j(z; G^B) \leq \Im_j(z; G^A) \forall z \in [0, z_{max}] \wedge \exists z | \Im_j(z; G^B) < \Im_j(z; G^A)$

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3. No dominance because A=B: $\Im_j(z; G^A) = \Im_j(z; G^B) \forall z \in [0, z_{max}]$
4. No dominance because A and B cross: $\exists z_1 \in [0, z_{max}] | \Im_j(z_1; G^A) < \Im_j(z_1; G^B) \wedge \exists z_2 \in [0, z_{max}] | \Im_j(z_2; G^B) > \Im_j(z_2; G^A)$

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The generic hypothesis subject to test is the following:

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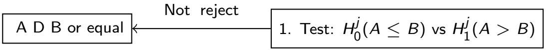
Let's define $H_0^j(A \leq B)$ and $H_1^j(A > B)$

The test of Barrett and Donald (2003)

1. Test: $H_0^j(A \leq B)$ vs $H_1^j(A > B)$

The test of Barrett and Donald (2003)



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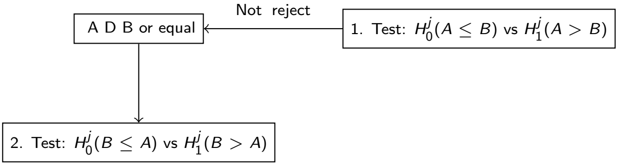

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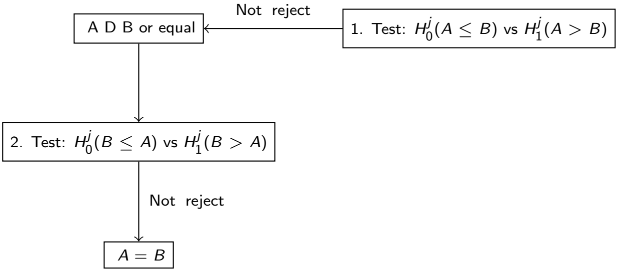

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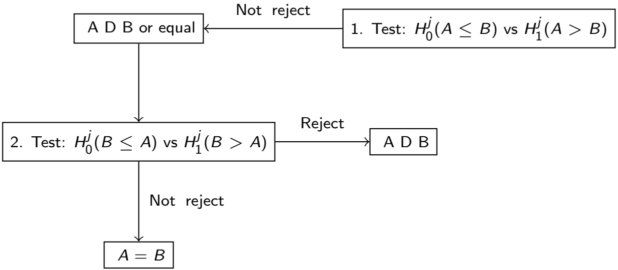

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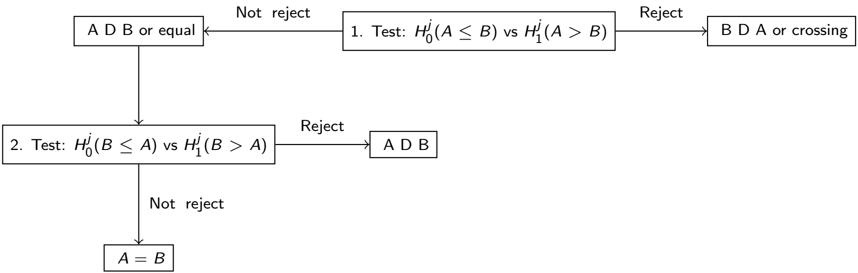

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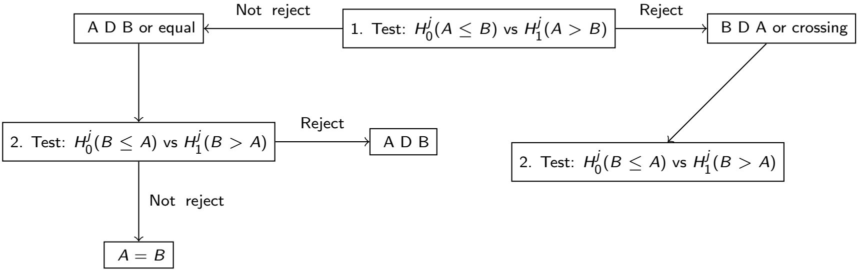

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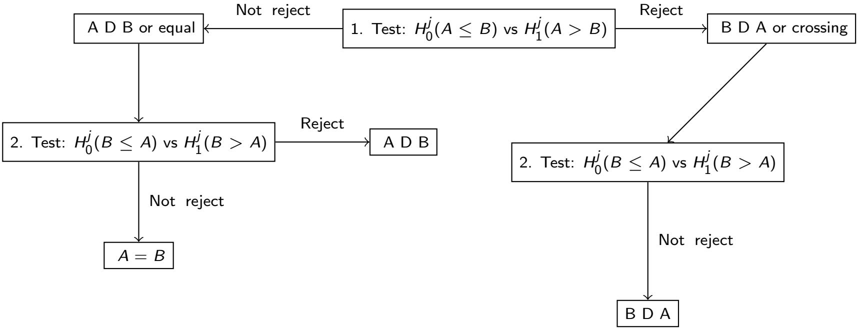

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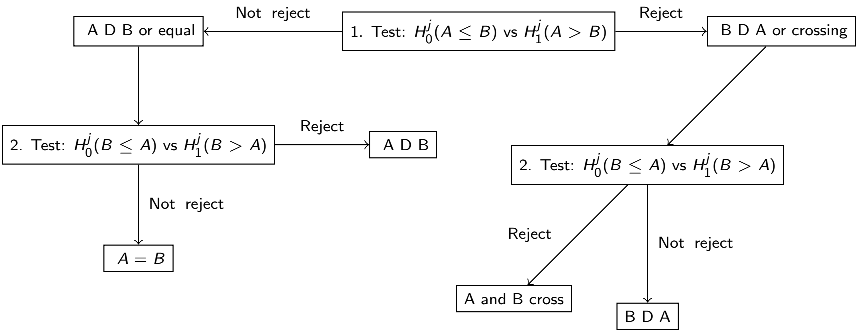

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- $\widehat{S}_j$ is the statistic we need. We now need to know how likely it is this value to appear under the null hypothesis
- There are different procedures to derive the distribution under the null hypothesis. Barrett and Donald (2003) develop two types. We are going to show one of them: a bootstrap method.

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3. The p-value is then approximated by: $p_j^{A,B} \simeq \frac{1}{R} \sum_{r=1}^{R} 1(\overline{S_j} > \widehat{S}_j)$
4. If, say, $p_j^{A,B} < 0.01$ we reject the null hypothesis: under the hypothesis a value like $\widehat{S}_j$ is very unlikely.

The test of Davidson and Duclos (2000)

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- They also have tests for cases where the poverty lines are absolute and cases where they are proportions of the mean or the median of the distributions. We will discuss the former (the latter requires techniques for the estimation of density functions)
- Also two cases are treated: dependent and independent samples (the latter, for instance, can come from a panel dataset or from a before and after tax survey)

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They go to show that the asymptotic covariance, $Ncov(\Im_j(z; \widehat{G^A}), \Im_j(z'; \widehat{G^B}))$ is:

$$\frac{1}{((s-1)!)^2} E((z - y^A)_+^{j-1}(z' - y^B)_+^{j-1}) - \Im_j(z; G^A)\Im_j(z'; G^B)$$

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In empirical applications $\Im_j(z; G^A)$ is replaced with $\Im_j(z; \widehat{G^A})$ and the expectation element is replaced with:

$$\frac{1}{N}\sum_{i=1}^{N}(z - y_i^A)_+^{j-1}(z' - y_i^B)_+^{j-1}$$

The statistic of Davidson and Duclos (2000): The case of two independent samples

In this case they calculate Z-statistic (distributed normal standard) for different values of the potential poverty lines, z:

$$Z_z^{A,B} = \frac{\Im_j(z; \widehat{G^A}) - \Im_j(z; \widehat{G^B})}{\sqrt{var(\Im_j(z; \widehat{G^A})) + var(\Im_j(z; \widehat{G^B}))}}$$

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where:

$$var(\Im_j(z;\widehat{G^A})) = \frac{1}{N_A((s-1)!)^2}\frac{1}{N_A}\sum_{i=1}^{N}(z-y_i^A)_+^{2(j-1)}$$

$$-\frac{1}{N_A}\Im_j(z;\widehat{G^A})^2$$

The statistic of Davidson and Duclos (2000): The case of two dependent samples

In this case the required $Z_z^{A,B}$ are:

$$\frac{\Im_j(z;\widehat{G^A}) - \Im_j(z;\widehat{G^B})}{\sqrt{var(\Im_j(z;\widehat{G^A})) - 2covar(\Im_j(z;\widehat{G^A}),\Im_j(z;\widehat{G^B})) + var(\Im_j(z;\widehat{G^B}))}}$$

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where the variances are calculated as before and the covariance is estimated as mentioned a few slides before too.

The statistic of Davidson and Duclos (2000): Using the statistics to make inference

Now, what do we do with all these Z-statistics estimated for several values of z across a wide plausible range?

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We are going to follow an approach of multiple contrasts followed by Anderson (1996) and Duclos et al. (2006) among others

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2. In term of the statistics this null hypothesis is rejected in favor of the specific alternative if and only if:

$$Z_z^{A,B} \leq -Z_{critical} \forall z \wedge \exists z | Z_z^{A,B} < Z_{critical}$$

3. Since we are performing multiple contrasts, we need to adjust the critical values accordingly. We use the values of the Studentized Maximum Modulus (SMM) distribution (Stoline and Ury, 1979).

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So for instance, failure to reject the null hypothesis would require the following conditions in terms of $Z_z^{A,B}$:

$$-Z_{critical} \leq Z_z^{A,B} \leq Z_{critical} \forall z$$