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Unidimensional Poverty Measurement

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Main Sources of this Lecture

- Foster and Sen (1997), Annexe of "On Economic Inequality".
- Foster (2006) "Poverty Indices"
- There are others: please see the readings list.

Sen (1976): Two-Steps

Poverty Measurement involves (at least) two steps. Actually there is also a prior step: where are we going to measure poverty? Here: income/consumption/expenditure.

1. **Identification:** *Who are the poor?*

   Dichotomises the population: poor/non-poor. Tool: **Poverty Line (z)**

Poor: $x_i < z$ **Non-Poor:** $x_i \geq z$

Types of Poverty Lines

- ***Absolute:*** Based on the cost of a set of goods and services considered necessary to have a satisfactory life. Usually a food poverty line (2100 calories a day) and a total poverty line are computed. (Cost of Basic Needs Method):

  Total Poverty Line=(1/Engel Coef.)*(Basic Food Basket)

- ***Relative:*** Defined with reference to the total income of the society. Examples: half of the median income. Problem: Confuses poverty with inequality.

- ***Hybrid Lines:*** Combinations of absolute and relative lines. Examples: $z=(z_r)^{\rho}(z_a)^{(1-\rho)}$ (Foster, 1998); $z=\max(z_a,\alpha+kM)$, with M being the median income, (Ravallion and Chen, 2009).

Sen (1976): Two-Steps

2. **Aggregation:** *How poor are we?*

   Construct an index that summarizes the data to form an overall picture of poverty.

- A poverty measure is a function P: D→R which, for each distribution *x* indicates the level *P(x;z)* of poverty in the distribution.

- We will adopt an absolute z approach and focus the discussion in terms of the indices.

Difference between inequality and poverty?

- Inequality measurement is intrinsically **relative**: comparing everyone's income with those of others.
- Poverty measurement is intrinsically **absolute**. It is deprivation vis-à-vis- an externally given poverty line z.

Axioms
(Classification of Foster, 2006)

- Invariance Axioms
- Dominance Axioms
- Continuity
- Subgroup Axioms (Consistency and Decomposability)

Invariance Axioms

x is obtained from y by a ***permutation*** of incomes if $x=Py$, where P is a permutation matrix.

Example: z=10, y=(4,8,9,15) x=(4,9,8,15)

1. *SYMMETRY (Anonymity): If* x is obtained from y by a ***permutation*** of incomes, then $P(x;z)=P(y;z)$.

Invariance Axioms

x is obtained from *y* by a ***replication*** if the incomes in *x* are simply the incomes in y repeated a finite number of times.

Example: z=10, y=(4,8,9,15)
x=(4,4,9,9,8,8,15,15)

2. *REPLICATION INVARIANCE (Population Principle):*
   If x is obtained from *y* by a ***replication***, then P*(x;z)=P(y;z)*.

Invariance Axioms

x is obtained from *y* by a ***an increment to a non-poor person*** if:

$$i)\, x_i > y_i \qquad for \qquad y_i \geq z$$

$$ii)\, x_j = y_j \qquad for\ all \qquad j \neq i$$

Example: z=10, y=(4,8,9,15) x=(4,9,8,16)

3. *Focus:* *If x* is obtained from *y* by ***an increment to a non-poor person***, then P*(x;z)=P(y;z)*.

Invariance Axioms

(x';z') is obtained from (x;z) by a ***proportional change*** if $(x';z')=(\alpha x;\ \alpha z)$ for $\alpha>0$

Example: z=10, x=(4,8,9,15);

z'=20, x'=(8,16,18,30)

4. *SCALE INVARIANCE (Zero-Degree Homogeneity):* If *(x';z')* is obtained from (x;z) by a ***proportional change***, then *P(x',z')=P(x,z)*.

Dominance Axioms

x is obtained from y by a ***decrement among the poor*** if for some *i*:

$$y_i < z$$

$$we \quad have \quad x_i < y_i$$

$$while \quad x_j = y_j \quad for\ all \quad j \neq i$$

Example: z=10, x=(4,8,9,15); x'=(4,7,9,15)

4. *MONOTONICITY: If x* is obtained from y by a decrement among the poor, then P*(x,z)>P(y,z)*.

Dominance Axioms

Given two distributions x and y, with the same mean. We say that x is obtained from y by a ***progressive transfer among the poor*** if for some i and j:

$$y_i < y_j < z$$

$$we \quad have \quad y_i < x_i \leq x_j < y_j$$

$$while \quad x_k = y_k \quad for\ all \quad k \neq i, j$$

Example: z=10, x=(4,8,9,15); x'=(5,7,9,15)

5. TRANSFER: *If x* is obtained from y by a progressive transfer among the poor, then P$(x,z)<P(y,z)$.

Dominance Axioms

- What if because an increase in the income of a poor, moves him/her above z? (the number of poor diminishes)

- What if because of a transfer a poor moves him/her above z? (the number of poor diminishes)

Dominance Axioms

x is obtained from y by an ***increment among the poor*** if for some *i*:

$$y_i < z$$

$$we \quad have \quad x_i > y_i$$

$$while \quad x_j = y_j \quad for\ all \quad j \neq i$$

Example: z=10, x=(4,8,9,15); x'=(4,8,11,15)

4'. Strong Monotonicity: If x is obtained from y by an increment among the poor, then P*(x,z)≤P(y,z).*

Dominance Axioms

Given two distributions x and y, with the same mean. We say that x is obtained from y by a ***regressive transfer among the poor*** if for some i and j:

$$y_i \leq y_j < z$$

$$we \qquad have \qquad x_i < y_i \leq y_j < x_j$$

$$while \qquad x_k = y_k \qquad for\ all \quad k \neq i, j$$

Example: z=10, y=(4,8,9,15); x=(4,6,11,15)

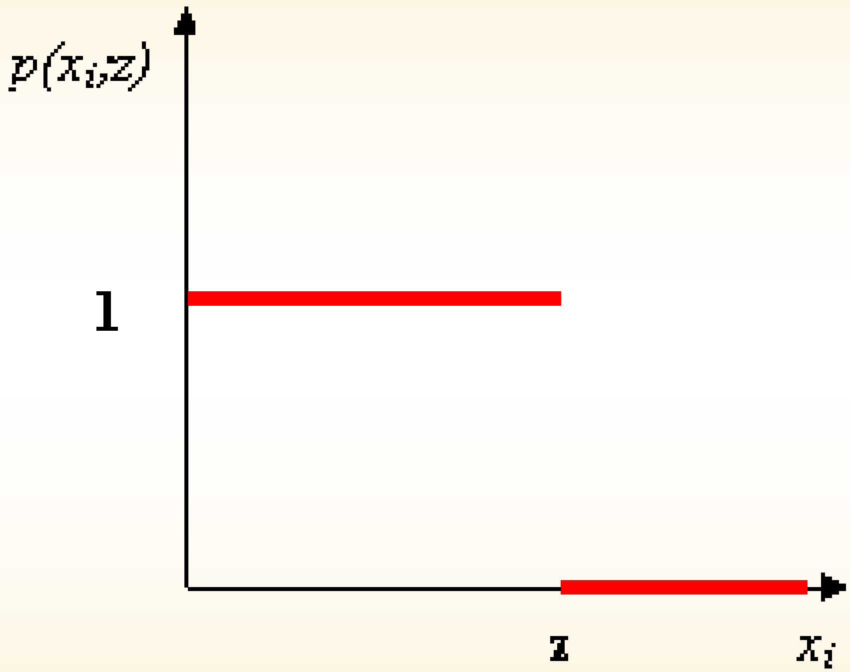
5'. STRONG TRANSFER (Sen, 1976): If x is obtained from y by a regressive transfer among the poor, then P*(x,z)>P(y,z)*.

Continuity

Restricted Continuity: For any sequence x^k having a fixed non-poor income distribution, if x^k converges to x, then $P(x^k;z)$ converges to $P(x;z)$.

Continuity: For any sequence x^k , if x^k converges to x, then $P(x^k;z)$ converges to $P(x;z)$.

Example of discontinuity in poverty measurement (the Headcount Ratio)



The Indicator Function
(base of the Headcount Ratio):

$$p(x_i;z) = \begin{cases} 1 & \text{if } x_i < z \\ 0 & \text{if } x_i \geq z \end{cases}$$

Continuity and Censoring

- Focus Axiom – Censored Distribution:
  - It is the distribution in which all incomes above z are replaced by z, and the other incomes remain unchanged: $x^{*}=x^{*}(z)$

$$x_i^* = x_i \quad \text{if} \quad x_i \le z$$

$$x_i^* = z \quad \text{if} \quad x_i > z$$

- For any continuous measure $P(x;z)$ satisfying the focus axiom: $P(x^{*};z)=P(x;z)$
- Censored distributions can also be used to create continuous measures from those satisfying restricted continuity.

Dominance Axioms

- Continuous poverty measures that satisfy invariance axioms satisfy monotonicity and transfer if and only if they satisfy the strong monotonicity and the strong transfer axioms. (Foster, 1984, Donaldson & Weymark, 1986).

Normalisation

If every individual has the poverty line income, then $P(x)=0$

Subgroup Axioms

- *SUBGROUP CONSISTENCY:* If P$(x';z)>P(x;z)$ and $P(y';z)=P(y;z)$, and $n(x')=n(x)$, $n(y')=n(y)$, then P$(x',y';z)>P(x,y;z)$.
- Example: z=10, x=(4,8,9,15). Suppose two groups: x_A=(4,8) x_B=(9,15).
- Say that poverty in A increases: x'_A=(3,8) while poverty in B remains the same: x'_B=(9,15).
- Then $P(x'_A)>P(x_A)$ and $P(x'_B)=P(x_B)$, and therefore one would expect: $P(x'_A, x'_B)>P(x_A, x_B)$, ie: P(3,8,9,15)>P(4,8,9,15).

Subgroup Consistency

- For which practical reason may it be important?
- Evaluation of poverty reduction programs!
- It can be seen as an extension of monotonicity:
  - Monotonicity requires poverty to fall when one person's poverty level is reduced. SC requires aggregate poverty to fall when one group's poverty level is reduced.

Subgroup Axioms

- *ADDITIVE SUBGROUP DECOMPOSABILITY:* A poverty measure P is decomposable if :

$$P(x,y)=\frac{n(x)}{n}P(x)+\frac{n(y)}{n}P(y)$$

(Extensible to any number of groups)

Then, one can calculate the contribution of each group to overall poverty:

$$C_X=(n(x)/n)P(x)/P(x,y)$$

Decomposability implies consistency.

The converse does not hold.

Subgroup Consistency & Additive Decomposability

P is a continuous, subgroup consistent poverty index if and only if P is a continuous, increasing transformation of a continous, decomposable poverty index.

(Foster and Shorrocks, 1991)

Other Axioms

- *Transfer Sensitivity:* If a transfer $t>0$ of income takes place from a poor person with income x_i to a poor person with income x_i+d, then the magnitude of the increase in poverty must be smaller for larger x_i. (Kakwani, 1980)

Poverty Measures

TYPES OF POVERTY MEASURES

Partial Measures

- Headcount Ratio
- Income Gap Ratio

Measures based on Poverty Gaps

- Foster Greer Thorbecke's family of measures (FGT)
- Sen's Measure

Measures based on Utility Gaps

- Watt's Measure
- Chakravarty's family of Measures
- Decomp. Version of Clark-Hemming & Ulph (CHU)'s Measure

Original CHU:

Based on Atkinson's EDE Income

Partial Poverty Indices: The Headcount Ratio (% of poor people)

$$H = q/n$$

- Example: x=(4,8,9,15); z=10 Then: H=3/4

  x'=(4,7,9,15) H=3/4

  x''=(3,9,9,15) H=3/4

- Insensitive to depth and distribution, ie: violates monotonicity and transfer. Satisfies only restricted continuity.
- Policy implication?

Partial Poverty Indices: Income Gap Ratio (average shortfall of the poor)

$$I = (z - \mu_p) / z$$

- Example: x=(4,8,9,15); z=10; μ_p=(4+8+9)/3=7; I=[(10-7)/10]=0.3
- Sensitive to depth, ie: satisfies monotonicity.
  x'=(4,7,9,15) I=[10-6.66]/10=0.33
- Insensitive to distribution: violates transfer.
  x''=(3,9,9,15) I=[10-7]/10=0.3
- Satisfies only restricted continuity.
- But has counterintuitive policy implications: the number of poor must not decrease!
  x'''=(4,8,11,13) I=[10-6]/10=0.4

Poverty measures based on poverty gaps

- **Individual Normalised Poverty Gap:**

Distance to the poverty line in poverty line units

$$g_i^* = (z - x_i^*) / z$$

(equivalent: $g_i=(z-x_i)/z$ if $x_i<z$ and $g_i=0$ othw.)

Example: x=(4,8,9,15); z=10

Gap for individual 1: $g_1=(10-4)/10=0.6$

He falls short 60% of the poverty line.

Poverty Indices: Sen's Measure (1976)

$$S(x;z) = \frac{1}{n}\sum_{i=1}^{n}\frac{(2r_i - 1)}{q} g_i^*$$

- r_i is the ranking of person *i* among the poor. The poorest person receives a r_i=q, the next poorest person receives a rank of r_i=q-1, and so on until the 'richest' poor receives r_i=1.
- It satisfies invariance and dominance axioms in the weak versions. Satisfies restricted continuity.
- It violates decomposability and subgroup consistency.

Poverty Indices: Sen's Measure (1976)

- Example: x=(4,8,9,15); z=10. Then q=3

$$S(x;z) = \frac{1}{n}\sum_{i=1}^{n}\frac{(2r_i - 1)}{q} g_i^*$$

x_i	r_i	(2ri-1)/q	gi	[(2ri-1)/q]gi
4	3	1.66	0.6	1
8	2	1	0.2	0.2
9	1	0.33	0.1	0.033
15	0	0	0	0
Sen's Measure				**0.31**

Poverty Indices:
Sen's Measure (1976)

- For n sufficiently big, Sen's Measure can be expressed as:

$$S(x;z) = H[I + (1 - I)G_p]$$

With G_p being the Gini Coefficient among the poor.

Poverty Indices: Sen's Measure

$$S(x;z) = H[I + (1 - I)G_p]$$

- Gini among the poor (Gp)

$$G(x) = \frac{2(4+5+1)}{(2)(3)^2(21/3)} = \frac{20}{126} = 0.16$$

- H=3/4
- I=0.3
- **S=0.75[0.3+(1-0.3)0.16]=0.31**

	4	**8**	**9**
4	0	4	5
8	4	0	1
9	5	1	0

Poverty Indices: Foster-Greer-Thorbecke 1984 (FGT)

$$FGT_{\alpha}(x;z) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{z - x_i^*}{z}\right)^{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\left(g_i^*\right)^{\alpha}$$

- Note that it is a family of measures, as parameter α can take different values.
- Alternative notation:
- Given $x=(x_1,x_2,\ldots,x_n)$. Define vector of alfa-normalised gaps: $g^{\alpha}=(g^{\alpha}{}_1,g^{\alpha}{}_2,\ldots,g^{\alpha}{}_n)$. Then:
- $FGT\alpha=\mu(g^{\alpha})$ (just the mean of the vector!)

Poverty Indices: FGT

- When α=0, FGT_0=H (Headcount Ratio)
- When α=1, FGT_1=HI=P_1
  (Per Capita Poverty Gap)
- When α=2, FGT_2=P_2
  (Per Capita Squared Poverty Gap)

Poverty Indices: FGT

- All FGT members satisfy invariance axioms and restricted continuity.
- When $\alpha>0$, FGTα satisfies continuity and strong monotonicity. Violate transfer.
- When $\alpha>1$, FGTα satisfies continuity, strong monotonicity and strong transfer.
- When $\alpha>2$, FGTα is also transfer sensitive, ie. distribution sensitive.

Example: FGT0
Headcount Ratio

- Example: $x=(4,8,9,15)$; $z=10$: $g^0=(1,1,1,0)$
- $P_0= \mu(g^0) =[1+1+1+0]/4=0.75$
- Insensitive to depth, ie: violates monotonicity.
  $x'=(4,7,9,15)$ $P_0= \mu(g^0) =[1+1+1+0]/4=0.75$
- Insensitive to distribution: violates transfer.
  $x''=(3,9,9,15)$ $P_0= \mu(g^0) =[1+1+1+0]/4=0.75$
- Violates continuity

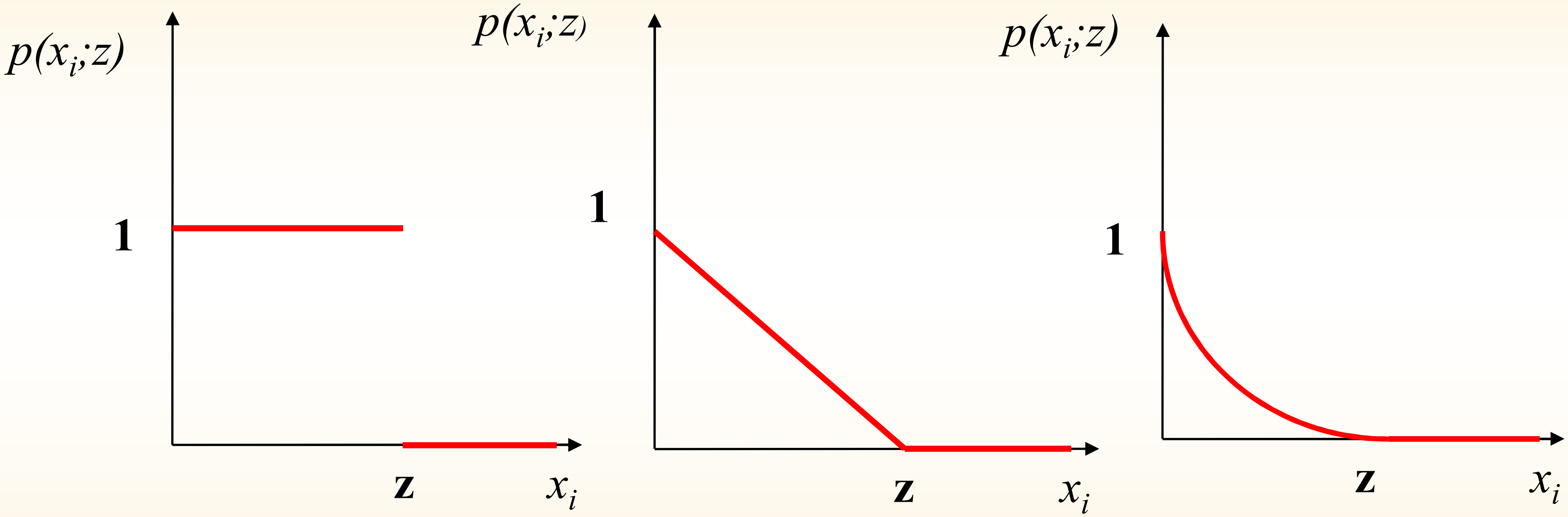
Example: FGT1
Per capita Poverty Gap

- Example: $x=(4,8,9,15)$; $z=10$: $g^1=(0.6,0.2,0.1,0)$
- $P_1= \mu(g^1) =[0.6+0.2+0.1+0]/4=0.225$
  (Also note that P1=HI Indeed: 0.225=0.75*0.3)
- Sensitive to depth, ie: satisfies monotonicity.
  $x'=(4,7,9,15)$ $P_1=[0.6+0.3+0.1+0]/4=0.25$
- Insensitive to distribution: violates transfer.
  $x''=(3,9,9,15)$ $P_1=[0.7+0.1+0.1+0]/4=0.225$
- Satisfies continuity
- Policy Implication?

Example: FGT2
Per capita Squared Poverty Gap

- Example: $x=(4,8,9,15)$; $z=10$: $g^2=(0.36,0.04,0.01,0)$
- $P_2=\mu(g^2)=[0.36+0.04+0.01+0]/4=0.1025$
- Sensitive to depth, ie: satisfies monotonicity.
  $x'=(4,7,9,15)$ $P_2=\mu(g^2)=[0.36+0.09+0.01+0]/4=0.115$
- Sensitive to distribution: satisfies transfer.
  $x''=(3,9,9,15)$ $P_2=\mu(g^2)=[0.49+0.01+0.01+0]/4=0.127$
- Satisfies continuity
- Policy Implication?

Individual Poverty Functions in a Graphically



The Indicator Function (base of the Headcount Ratio):
$p(x_i;z)=1$ if $xi<z$
0 if $xi\geq z$

The (Individual) Normalized Poverty Gap:
$p(x_i;z)=[(z-x_i)/z]$

The (Individual) Squared Normalized Poverty Gap:
$p(x_i;z)=[(z-x_i)/z]^2$

Poverty Indices

- Foster Greer and Thorbecke (1984)-FGT or Pα
- For n sufficiently big, P_2 can also be expressed as:

$$P_2 = H[I^2 + (1 - I)^2 C_p^2]$$

- With C_p^2 being the Squared Coefficient of Variation among the poor.

Poverty Indices: FGT

$$P_2 = H[I^2 + (1 - I)^2 C_p^2]$$

- Example:

$x=(4,8,9,15)$; $z=10$

$CV^2p=((1/7)[(7-4)^2+(8-7)^2+(9-7)^2]^{1/2})^2=0.095$

$FGT_2=0.75[(0.3)^2+(1-0.3)^2(0.095)]=0.1025$

FGT & Sen's Measures

- Both based on poverty gaps g_i
- Both are sensitive to the number of poor (H), the depth of their poverty (I) and the distribution of income among the poor.

- Key differences:

✓ **Continuity** (P_2 is cont while S is not)-This can be solved using the censored distribution.

✓ **Subgroup consistency and decomposability** (P_2 satisfies both while S violates both): This is a consequence of the inequality measure they are related to: Gini vs. CV^2.

Subgroup Consistency and Decomposability – pros and cons

- ***Pros:***
  - Allow breaking-down poverty into its constituent components (groups that contribute more).
  - Evaluation of poverty reduction programs: it would be puzzling to have the overall poverty level going up while poverty in each subgroup goes down!
- ***Cons:***
  - One may consider that interdependence matters, and that one's poverty may depend not only on her own shortfall to z but also on her shortfall *vis-à-vis* the shortfall of others (ie: her <u>relative</u> position to others), then subgroups consistency and decomposability may not be such key properties.

Poverty Indices: Clark, Hemming and Ulph (1981)- CHU

$$C_\beta(x;z) = \begin{cases} 1 - \left[\dfrac{1}{n} \sum_{i=1}^{n} \left(\dfrac{x_i^*}{z} \right)^{\beta} \right]^{1/\beta} & \beta \le 1, \beta \ne 0 \\ 1 - \left[\dfrac{1}{n} \prod_{i=1}^{n} \left(\dfrac{x_i^*}{z} \right) \right]^{1/n} & \beta = 0 \end{cases}$$

- All incomes are normalised by the poverty line
- Takes Atkinson's EDE income (general mean with $\beta<1$) among the poor, normalised by the poverty line and compares that to the 'reference' 1.
- When $\beta = 1$, the measure is just the per capita poverty gap. It is just taking the normalised mean income among the poor and subtracting it from 1.
- When $\beta < 1$, it penalises any inequality among the poor (the lower will be the normalised general mean income with respect to the mean).

Poverty Indices: CHU

- Only positive incomes are allowed.
- It satisfies invariance axioms, dominance axioms (for $\beta<1$) and continuity.
- Each member of the family is not decomposable (except for $\beta=1$) but it is subgroup consistent.
- β is a measure of ‘aversion to inequality in poverty’. The lower, the higher is the aversion to inequality.

Poverty Indices based on utility gaps

- Dalton type of poverty measures (Hagenaars 1987): The utility loss due to poverty. Utility shortfalls.

$$D(x;z) = A(z)\frac{1}{n}\sum_{i=1}^{n} u(z) - u(x_i^*)$$

- A(z) is a normalisation factor.

Poverty Indices: Watt's (1969) Index

$$W(x;z) = \frac{1}{n}\sum_{i=1}^{n}(\ln z - \ln x_i^*)$$

- Can be interpreted as a Dalton type of poverty measure, where $u(x_i)=\ln(x_i)$, $A(z)=1$.
- Only positive incomes are allowed.
- Satisfies invariance axioms, monotonicity, continuity and decomposability.

Poverty Indices: Chakravarty's (1983) Index

$$Ch(x;z)=\frac{1}{n}\sum_{i=1}^{n}\left[1-\left(\frac{x_i^*}{z}\right)^{\beta}\right] \qquad 0<\beta<1$$

- Can be interpreted as a Dalton type of poverty measure, where $u(x_i)=(x_i)^{\beta}$ and $A(z)= 1/z^{\beta}$
- Satisfies all invariance, dominance, continuity and decomposability axioms.

Poverty Indices: Decomposable CHU (Atkinson, 1987)

$$DC_\beta(x;z) = \begin{cases} \dfrac{1}{\beta n}\sum\left[1-\left(\dfrac{x_i^*}{z}\right)^\beta\right] & \beta \le 1, \beta \ne 0 \\ \dfrac{1}{n}\sum(\ln z - \ln x_i^*) & \beta = 0 \end{cases}$$

- Note that the first line is an extension of Chakravarty's Index and the second line is Watt's Index.
- Can be interpreted as a Dalton type of poverty measure, where $u(x_i)=(x_i)^\beta/\beta$ and $A(z)=1/(z^\beta)$ for $\beta \neq 0$ and $u(x_i)=\ln(x_i)$ for $\beta=0$.
- Satisfies all invariance, dominance, continuity and decomposability axioms.