Wu-Hausman Exogeneity tests

Hausman Test in general:

Procedure: H_0: null, H_a: alternative

$\hat{\beta}$: consistent and efficient under H_0
inconsistent under H_a

$\tilde{\beta}$: consistent under both H_0 and H_a
but not efficient under H_0

Then

$$(\tilde{\beta} - \hat{\beta})' \left[V(\tilde{\beta}) - V(\hat{\beta})\right]^{-1} (\tilde{\beta} - \hat{\beta}) \sim \chi^2_K$$

K: no. of parameters in β

For checking whether a particular set of variables is exogenous in a regression model say to check if X_2 is exogenous in

$$y = X_1 \beta_1 + X_2 \beta_2 + \varepsilon \qquad (1)$$

we have (for H_0: X_2 exogenous)

$\hat{\beta}_{OLS}$: consistent and efficient under H_0

$\hat{\beta}_{IV}$: consistent under H_a

$$\Rightarrow (\hat{\beta}_{IV} - \hat{\beta}_{OLS})' \left[V(\hat{\beta}_{IV}) - V(\hat{\beta}_{OLS})\right]^{-1} (\hat{\beta}_{IV} - \hat{\beta}_{OLS}) \sim \chi^2_{K_2}$$

(K_2 only because some elements of the difference become zero, X_1 being its own instrument)

But, it can be shown (see Davidson and MacKinnon (2004)) that this test is <u>equivalent</u> to testing $\delta = 0$ in the regression

$$y = X\beta + \hat{V}_2 \delta + \text{residuals} \qquad (2)$$

where $\hat{V}_2 = X_2 - W(W'W)^{-1}W'X_2$ are the residuals of an auxiliary regression of X_2 on W, the instrument matrix for X

So the test statistics is

$$\hat{\delta}_{ols}' \left(V(\hat{\delta}) \right)^{-1} \hat{\delta}_{ols} \sim \chi^2_{K_2}$$

<u>Remark</u>

OLS of β in (2) is <u>identical</u> to IV of β in (1) with W instrumenting X

<u>Simultaneous Equation Context</u>

Test $\delta = 0$ in

$$y_1 = Y_1 \gamma_1 + X_1 \beta_1 + \hat{V}_1 \delta + u$$

where $\hat{V}_1 = Y_1 - X(X'X)^{-1}X'Y_1 = Y_1 - X\hat{\Pi}_1$