Robustness analysis with the AF measures

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$$H^A(k) \leqq H^B(k) \forall k \in [1, D] \leftrightarrow F^A(c) \geqq F^B(c) \forall c \in [0, D]$$

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1. The dimension-specific poverty lines (i.e. the "first" cut-off): z_d
2. The weights attached to every variable/dimension: w_d
3. The value that the weighted sum of deprivations need to surpass in order to identify someone as poor (i.e. the "second" cut-off): k

Statistical tools to address the robustness of rankings generated by the AF measures

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   - Another one: Alkire and Foster (2009) and Lasso de la Vega (2009) derive dominance conditions over k keeping weights and lines fixed
2. To derive conditions under which a ranking is robust regardless of lines, weights and multidimensional counting thresholds (this is harder but still doable)

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- Some basic (first-order) stochastic dominance conditions for the M0 and H involving multidimensional thresholds, weights and lines.
- Some basic robustness tests for weights.

The counting vector: key ingredient for dominance conditions for M0 and H

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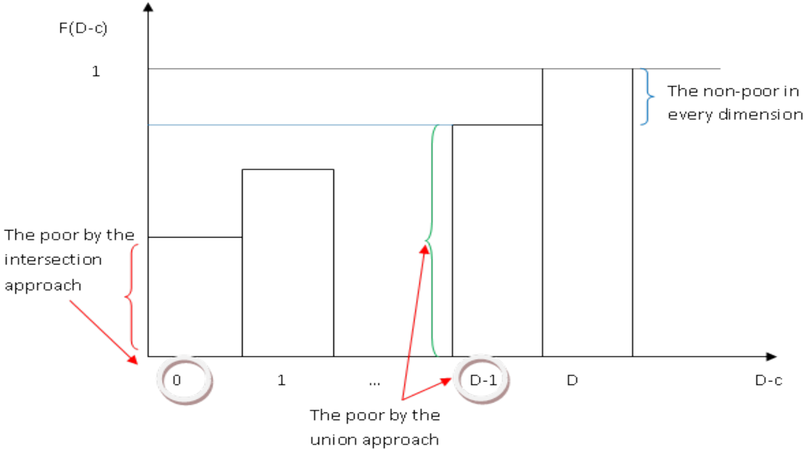
Then consider a distribution of deprivations, $D - c$, in the population that can take values from 0 (poor in every dimension) to D (non-poor in every dimention). A typical cumulative distribution is:

A typical cumulative distribution of D-c



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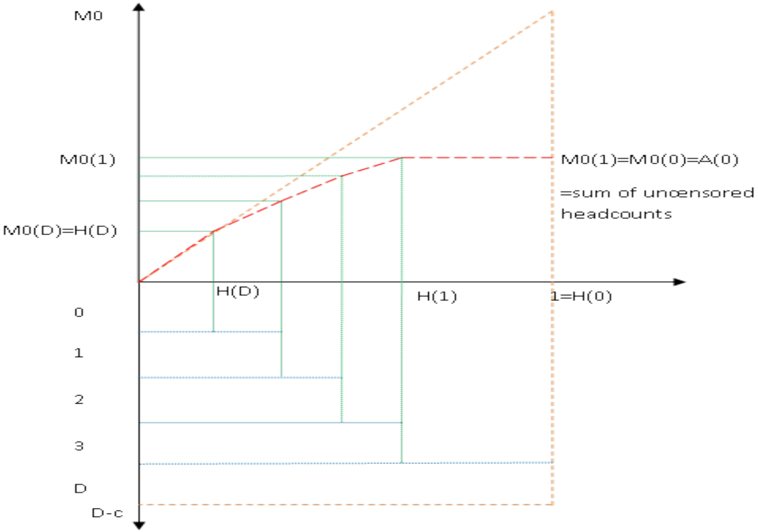
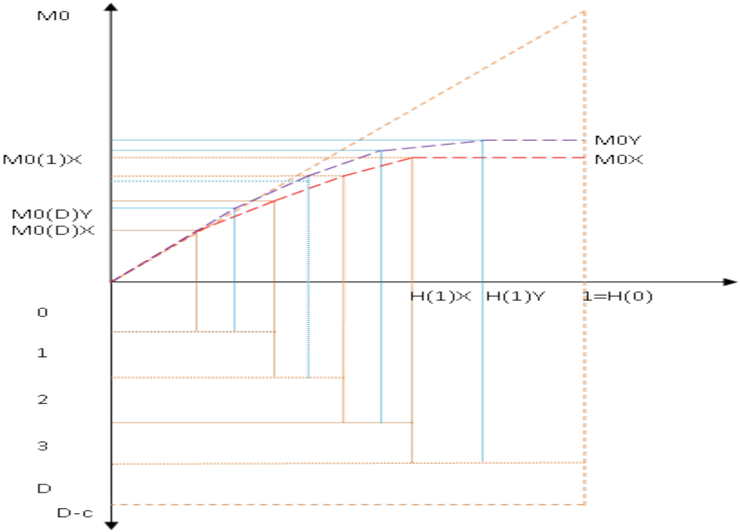
$$F^A(D-c) \leqq F^B(D-c) \forall (D-c) \in [0, D] \leftrightarrow H^A \leqq H^B \forall (D-c) \in [0, D]$$

$$H^A \leqq H^B \forall (D-c) \in [0, D] \rightarrow M^A \leqq M^B \forall (D-c) \in [0, D]$$

The key dominance result in a pair of graphs: I



The key dominance result in a pair of graphs: II



Proof: Alkire and Foster explained

Notice that $M0$ can be expressed in terms of H the following way:

$$M0(k) = \frac{1}{D}[H(D)D + \sum_{j=k}^{D-1} j[H(j) - H(j+1)]]$$

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Therefore $H^A(k) \leqq H^B(k) \forall k \in [1, D] \rightarrow M^A(k) \leqq M^B(k) \forall k \in [1, D]$

Further dominance results: now incorporating weights and poverty lines

We saw that for $M^A(k) \leqq M^B(k) \forall k \in [1, D]$ to hold we need:
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Are there any conditions that ensure that the latter holds, in turn, for any weights and poverty lines?

Yes, it is work in progress, but the condition seems to be the following:

$$\overline{F^A}(x_1, ...x_D) \geqq \overline{F^B}(x_1, ...x_D) \forall (x_1, ...x_D) \in [x_{1,min}, x_{1,max}] ... \times ... [x_{D,min}, x_{D,max}]$$

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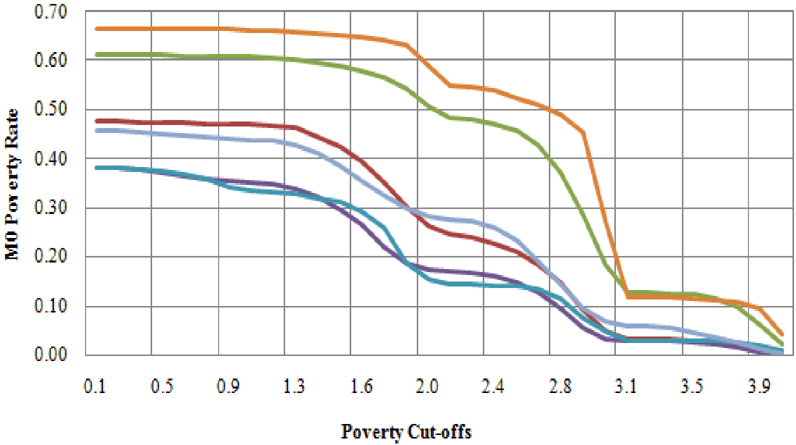
Example: Test of dominance across six African countries by Batana



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- We need some other ways to understand how robust are the rankings to changes in weights and cut-offs
- We may not always need to check dominance for the entire distribution
  - Smaller sample size for extreme values of the cut-off

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- Second cut-off is denoted by $k \in \{1, 2, \ldots, D\}$
- Let us denote the rank of N countries by the column vector

$$R = (R_1, R_2, \ldots, R_N)$$

We assume $R_1 < R_2 < \ldots < R_N$

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- If $R_n = R'_n$ for all $n = 1, \ldots, N$ then the ranking is completely robust with respect to this alternative specification
- However, if it is not then we need to find a method to check the robustness of ranking

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- Then

$$\tau = \frac{C - D}{C + D}$$

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Empirical Illustration

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Cut-off (k)	**3**	**4**	**5**	**6**	**7**
4	1.00	-	-	-	-
5	0.99	1.00	-	-	-
6	0.99	1.00	1.00	-	-
7	0.97	0.97	0.98	0.98	-
8	0.96	0.96	0.96	0.97	0.98

Empirical Illustration: Alkire ans Santos 2010

			MPI 1	MPI 2	MPI 3	MPI 4
			Excluding Enrolment	Using weight-for-age Selected Measure	Using weight-for-height	Using height-for-age
MPI 2	Using weight-for-age (Selected Measure)	Pearson	0.989			
		Spearman	0.988			
		Kendall (Taub)	0.920			
MPI 3	Using weight-for-height	Pearson	0.986	0.996		
		Spearman	0.985	0.999		
		Kendall (Taub)	0.908	0.984		
MPI 4	Using height-for-age	Pearson	0.987	0.998	0.996	
		Spearman	0.987	0.998	0.996	
		Kendall (Taub)	0.917	0.969	0.962	
MPI 5	Using under 5 mortality (rather than age non-specific mortality)	Pearson	0.991	0.998	0.997	0.996
		Spearman	0.989	0.997	0.995	0.996
		Kendall (Taub)	0.920	0.975	0.966	0.959
Number of countries:		**51 (All DHS and three MICS countries which have Birth History)**				