Robustness in Weighting

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This Lecture

- Introducing Composite Indices (CIs)
  - Linear composite indices
- Introducing the set of all possible weights
- Checking robustness of comparisons using CIs with respect to the chosen weighting scheme
- Proposing a measure of robustness
- Prevalence of comparisons
- Robustness Versus Association among dimensions

Composite Indices (Linear)

Many multidimensional indices take the form

$$C(\mathbf{x};\mathbf{w}) = \mathbf{w}\cdot\mathbf{x} = \Sigma w_d x_d$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_D)$$

is a vector of dimensional achievements and

$$\mathbf{w} = (w_1, w_2, \ldots, w_D)$$

is a vector of weights satisfying

$$w_1, \ldots, w_D \geq 0 \text{ and } w_1 + \cdots + w_D = 1$$

"Composite Indicator"

Examples

- Human Development Index or HDI (UNDP)
- Gender Empowerment Measure or GEM (UNDP)
- Global Peace Index (Inst. Econ Peace)
- Environmental Sustainability Index (Yale)
- Child Well-being Index (Inst. Child Dev.)
- Human Poverty Index (UNDP) Monotone Transform
- Alkire & Foster Poverty Indices (OPHI) Union Approach

Why this Form?

- Natural
- Easy to understand
- Statistical properties

Key challenge…how to choose weights?

Choice of Weights

Methods

- Normative
- Statistical e.g. Principal component analysis
- Equal weights

“Our argument for equal indicator weights is based on the premise that no objective mechanism exists to determine the relative importance of the different aspects of environmental sustainability.”

Environmental Sustainability Index

“…we have no reliable basis for doing [otherwise]”

Mayer and Jencks, 1989

Note: Analogous to “Principle of insufficient reason”

Choice of Weights

Note

- Inevitable arbitrariness in choice of weights
- Arrow's critique of HDI's weights w = (1/3,1/3,1/3)
  - *Amartya Sen: A life reexamined* (*documentary on Sen's life produced by Suman Ghosh, 2003*)

Why important?

- Comparison could be ambiguous or robust

Example 1

Country	HDI
Ireland (x)	0.956
Canada (y)	0.950

$w^0 = (1/3, 1/3, 1/3)$

Country	HDI
Ireland (x)	0.937
Canada (y)	0.942

$w = (1/2, 1/4, 1/4)$

$$C(x;w^0) > C(y;w^0)$$

at initial weight w^0 and yet

$$C(x;w) < C(y;w)$$

at some other reasonable w

"Ambiguous"

Example 2

Country	HDI
Australia (x)	0.957
Sweden (y)	0.951

Same ranking for all w

$$C(x;w^0) > C(y;w^0)$$

at initial w^0 and

$$C(x;w) \geq C(y;w)$$

at all reasonable w

"Robust"

How to discern?

Clearly,

Ranking $C(x;w^0) > C(y;w^0)$ reveals nothing about the robustness or ambiguity of the comparison

The following look the same – but are different

Country	HDI
Ireland (x)	0.956
Canada (y)	0.950

Country	HDI
Australia (x)	0.957
Sweden (y)	0.951

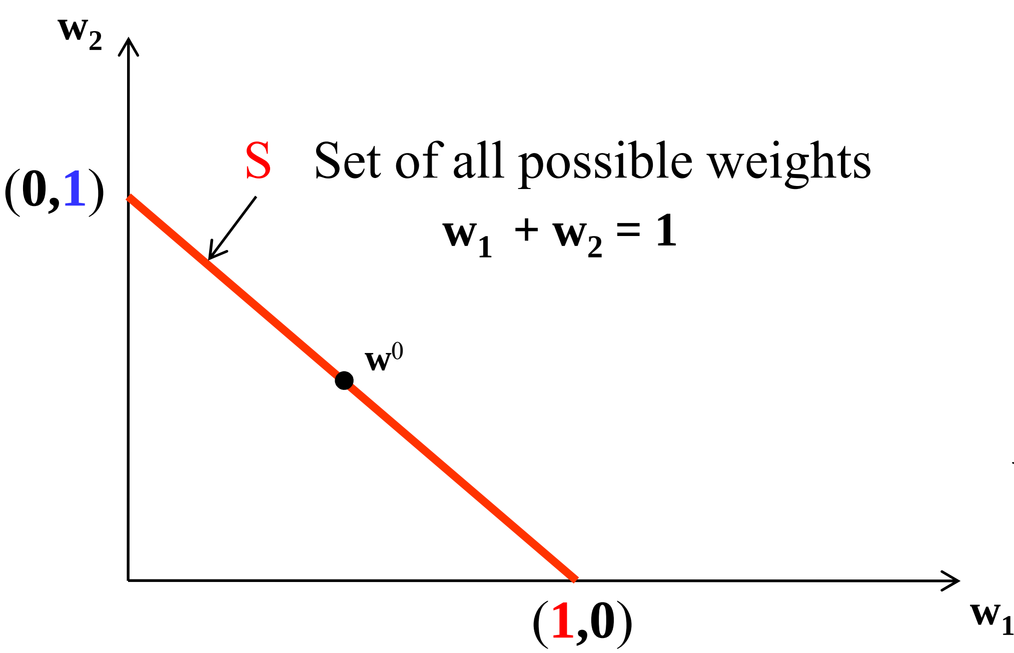
Question: How should we distinguish these seemingly analogous robust and ambiguous comparisons?

Two Approaches

1. Sensitivity analysis
   - Calculates the confidence interval of each index depending on how different scenarios change
   - If the confidence intervals of two indices intersect, their ranking can be stated to be ambiguous
     - Saisana, Saltelli, and Tarantola (2005)
2. Robustness Approach
   - Directly compares two units for varying weights
     - Cherchye, Ooghe, and Puyenbroeck (2008)
     - ✓ Foster, McGillivray, and Seth (2009)

How to Summarize the Set of All Possible Weights?

- Case I: Two Dimensions



Definitions

S a 2-dimensional simplex

Summarizes all possible weight

w^0 Initial weighting vector

Example

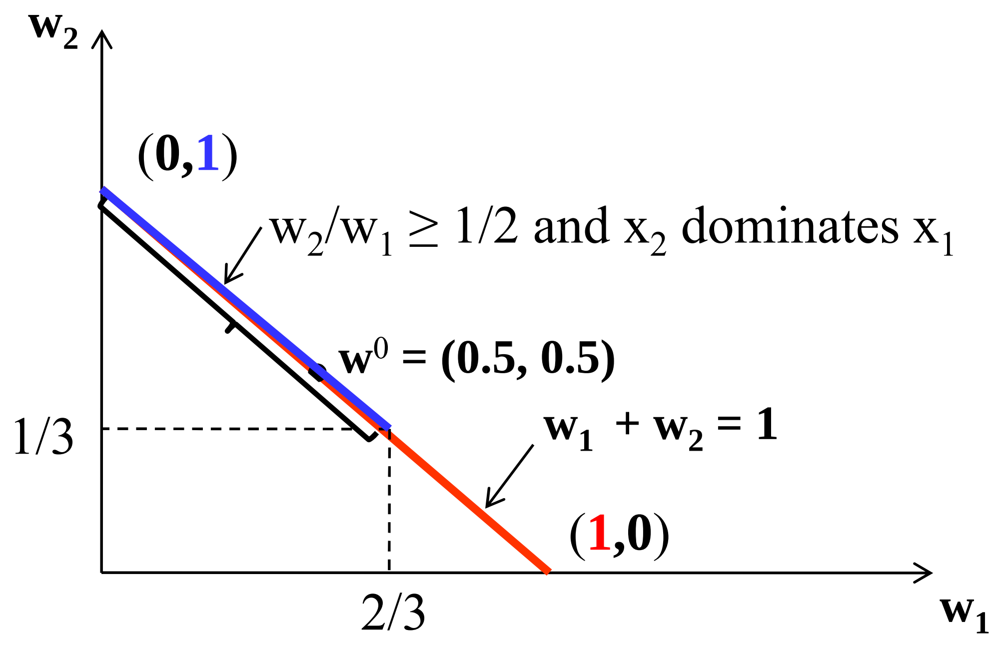
- Suppose two vectors are $x_1 = (0.5, 0.5)$ and $x_2 = (0.4, 0.7)$, and the initial weighting vector is $w^0 = (0.5, 0.5)$
- Thus, $C(x_2;w^0) \geq C(x_1;w^0)$
  - What about other possible weights?
- x_2 is better than x_1 for all weights $w' = (w_1,w_2)$ such that

$$0.4w_1 + 0.7w_2 \geq 0.5w_1 + 0.5w_2$$

$$\Rightarrow 0.2w_2 - 0.1w_1 \geq 0$$

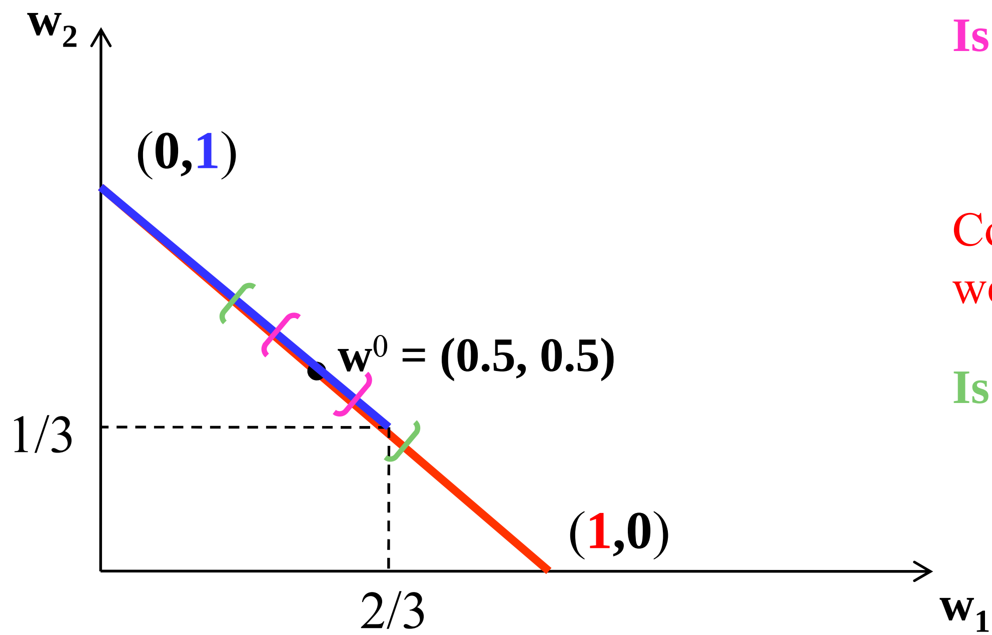
$$\Rightarrow w_2/w_1 \geq 1/2$$

Who Dominates?



Question – What other weights around the initial weight w^0 enable x_2 to be higher than x_1?

Approach (with Two Dimensions)



Is the comparison robust for W?

- **Yes.**

Consider another set of reasonable weighting vectors around w^0, W'.

Is the comparison robust for W'?

- **No!**

Define

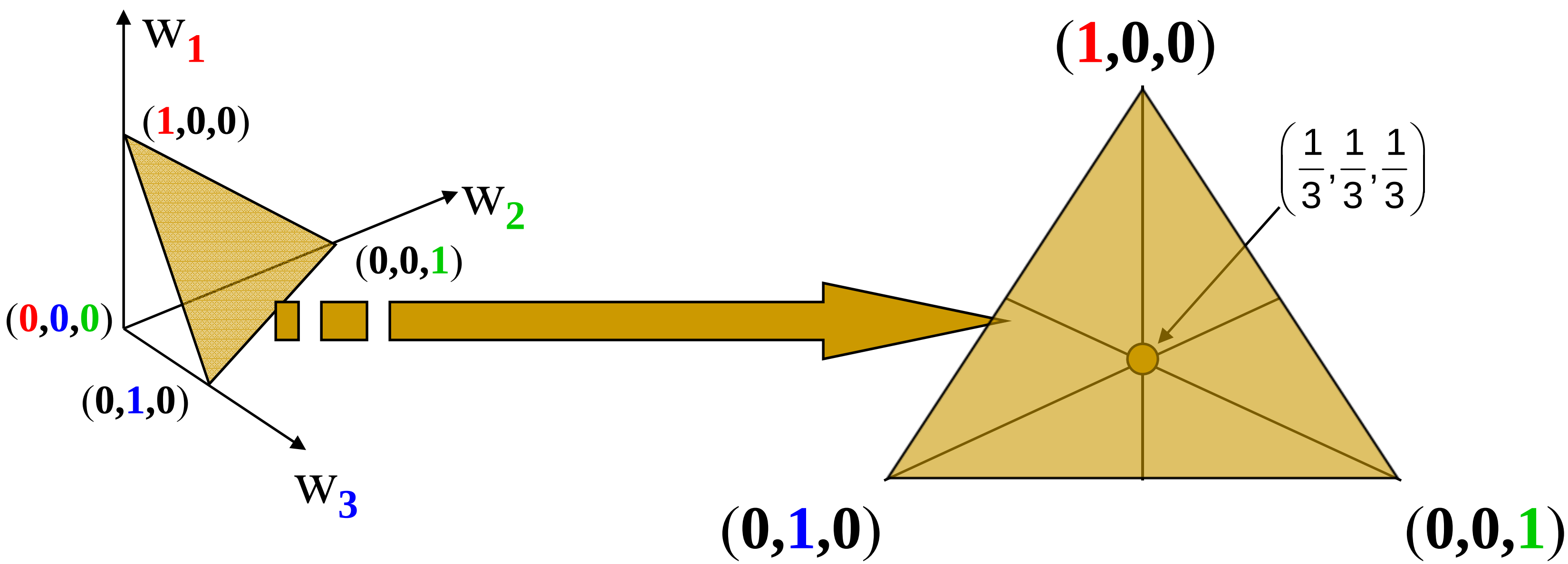
W Set of "reasonable" weighting vectors around w^0

Shape of W?

- We do not know yet.

How to Summarize the Set of All Possible Weights?

- Case II: Three Dimensions (e.g. HDI)



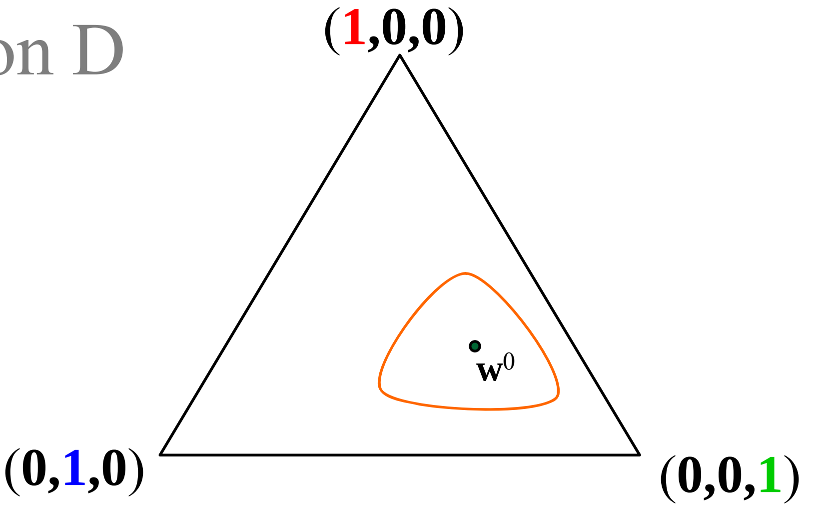
$w_1 + w_2 + w_3 = 1$ & $w_1, w_2, w_3 \geq 0$

At (1,0,0), HDI $= 1^* x_1 + 0^* x_2 + 0^* x_3 = x_1$

Approach (with Three Dimensions)

Definitions

S Simplex of dimension D



w^0 Initial weighting vector

W Set of "reasonable" weighting vectors around w^0

Shape of W?

– We do not know yet.

Approach

Robustness of ranking

$x \; C_W \; y$ if and only if $C(x;w^0) > C(y;w^0)$ and

$C(x;w) \geq C(y;w)$ for all $w \in W$

Analogous constructs

Partial comparability (Sen, 1970)

Poverty ordering (Foster-Shorrocks, 1988)

Q/ Which W?

A/ We use nested sets W_r where $r \in [0,1]$

Contamination model of ambiguity (Ellsburg, 1961)

The C_0 Relation

- Suppose W = $\{w^0\} = W_0$ (A set with a single element – the initial weighting vector)
  - Denote resulting relation by C_0 so that
    - x C_0 y if and only if $C(x;w^0) > C(y;w^0)$
  - Original complete ordering of the composite index
- Interpretation of W_0?
  - Supremely confident in choice of initial weightings - offers no robustness test at all
  - **Implausible!!**

HDI Top Ten According to C_0

Rank	Country	HDI
1	Norway	0.965
2	Iceland	0.960
3	Australia	0.957
4	Ireland	0.956
5	Sweden	0.951
6	Canada	0.950
7	Japan	0.949
8	United States	0.948
9	Switzerland	0.947
10	Netherlands	0.947

Human Development Report 2006, UNDP

Complete Ordering

Coun.		**Nor**	**Ice**	**Aus**	**Ire**	**Swe**	**Can**	**Jap**	**USA**	**Swit**	**Neth**
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2	C_0									
Aus	3	C_0	C_0								
Ire	4	C_0	C_0	C_0							
Swe	5	C_0	C_0	C_0	C_0						
Can	6	C_0	C_0	C_0	C_0	C_0					
Jap	7	C_0	C_0	C_0	C_0	C_0	C_0				
USA	8	C_0	C_0	C_0	C_0	C_0	C_0	C_0			
Swit	9	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0		
Neth	10	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	

Column Dominates Row

The C_1 Relation

Suppose W = S = W_1 (The entire simplex of all weighting vectors)

Denote resulting ranking by C_1 so that

x C_1 y if and only if $C(x;w^0) > C(y;w^0)$ and

$$C(x;w) \geq C(y;w) \text{ for all } w \in S$$

Interpretation of W_1?

No confidence in choice of initial weightings – offers full robustness test

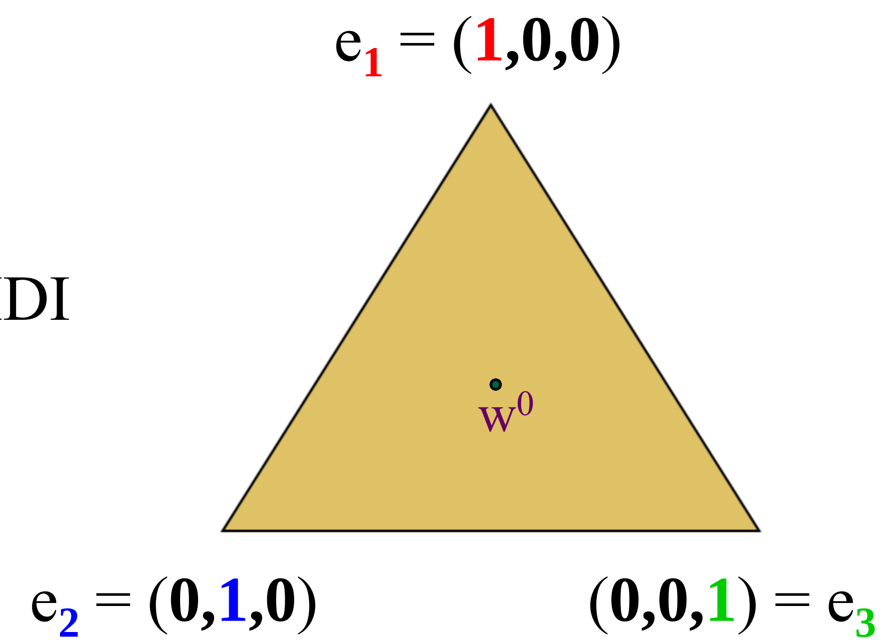
Stringent requirement!!

Characterization of C_1?

Simplex S

$C(x;w^0) = (x_1+x_2+x_3)/3 = \text{HDI}$

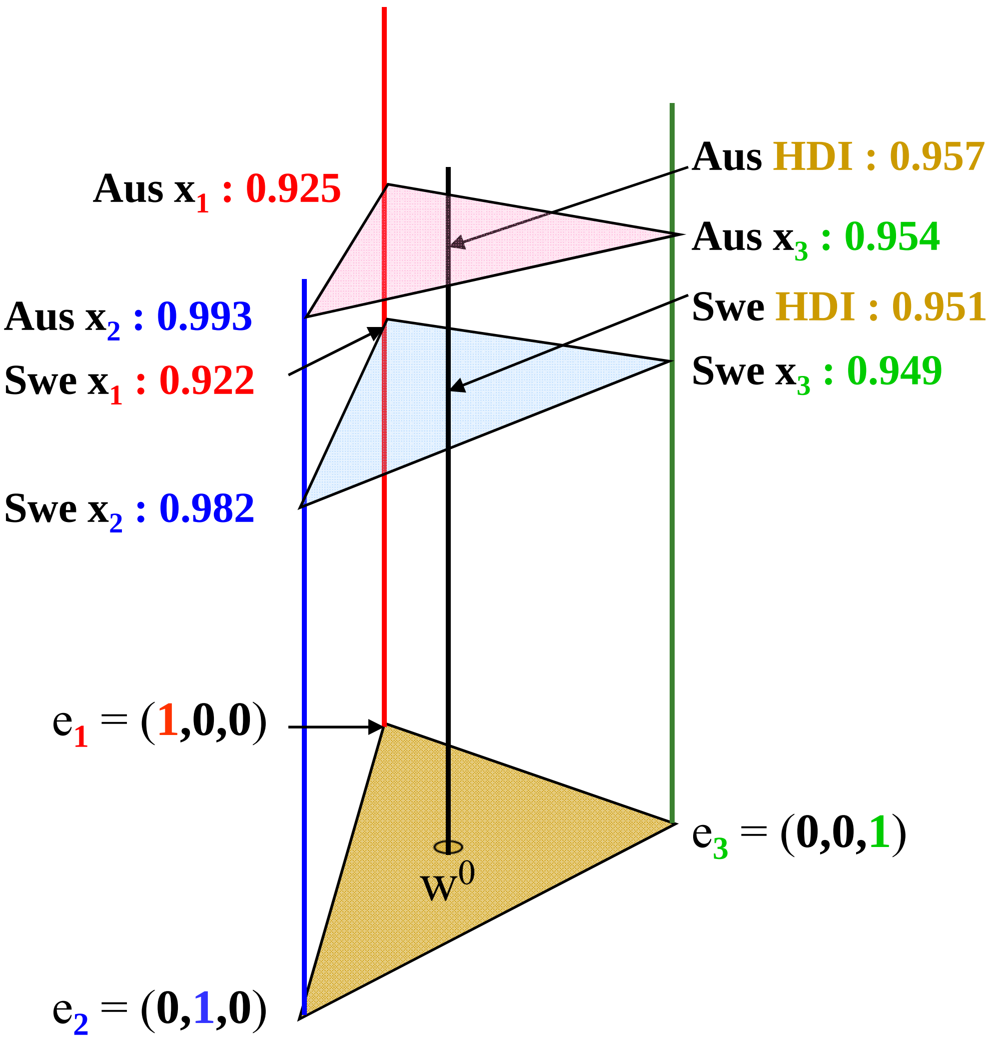
$C(x;e_i) = x_i$; $i = 1,2,3$



Under what condition, one country dominates another country for all weighting vector in S?

Theorem 1: $x\ C_1\ y$ if and only if $x \geq y$ and $x \neq y$

Example 1

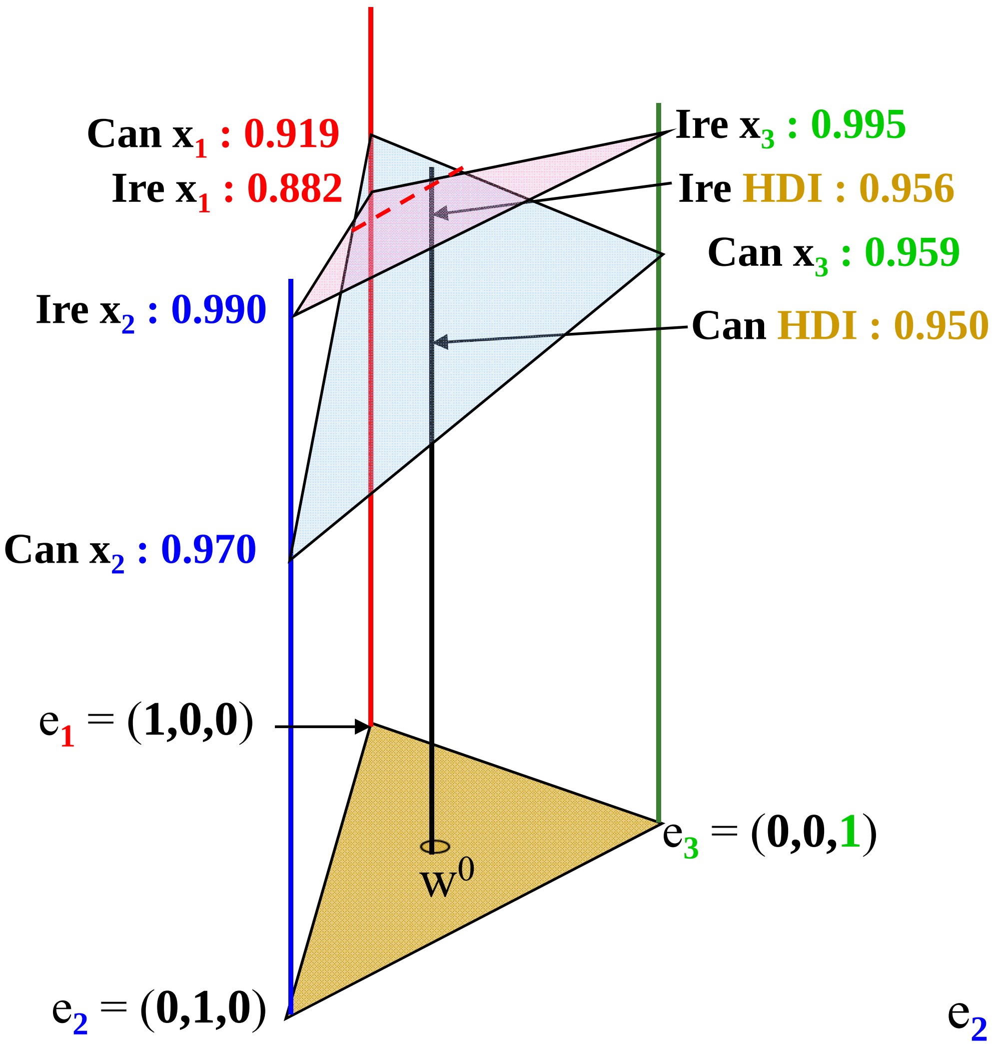


Coun.	HDI	x_1	x_2	x_3
Aus	0.957	0.925	0.993	0.954
Swe	0.951	0.922	0.982	0.949

Aus C_1 Swe

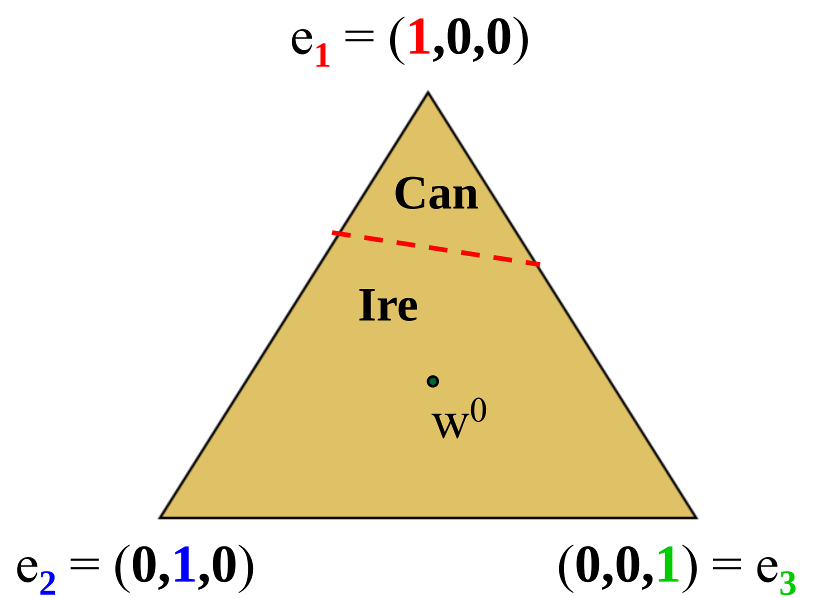
Theorem 1: $x\ C_1\ y$ if and only if $x \geq y$ and $x \neq y$

Example 2



Coun.	HDI	x_1	x_2	x_3
Ire	0.956	0.882	0.990	0.995
Can	0.950	0.919	0.970	0.959

Ireland Canada ranking is not fully robust.



Recall C_0 Ranking

Coun.		**Nor**	**Ice**	**Aus**	**Ire**	**Swe**	**Can**	**Jap**	**USA**	**Swit**	**Neth**
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2	C_0									
Aus	3	C_0	C_0								
Ire	4	C_0	C_0	C_0							
Swe	5	C_0	C_0	C_0	C_0						
Can	6	C_0	C_0	C_0	C_0	C_0					
Jap	7	C_0	C_0	C_0	C_0	C_0	C_0				
USA	8	C_0	C_0	C_0	C_0	C_0	C_0	C_0			
Swit	9	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0		
Neth	10	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	

Column Dominates Row

Fully Robust Ranking C_1???

Coun.		Nor	Ice	Aus	Ire	Swe	Can	Jap	USA	Swit	Neth
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2	C_0									
Aus	3	C_0	C_0								
Ire	4	C_0	C_0	C_0							
Swe	5	C_0	C_0	C_0	C_0						
Can	6	C_0	C_0	C_0	C_0	C_0					
Jap	7	C_0	C_0	C_0	C_0	C_0	C_0				
USA	8	C_0	C_0	C_0	C_0	C_0	C_0	C_0			
Swit	9	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0		
Neth	10	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	C_0	

Column Dominates Row

Fully Robust Ranking C_1???

Coun.		**Nor**	**Ice**	**Aus**	**Ire**	**Swe**	**Can**	**Jap**	**USA**	**Swit**	**Neth**
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2										
Aus	3										
Ire	4										
Swe	5			C_1							
Can	6		C_1								
Jap	7										
USA	8										
Swit	9		C_1								
Neth	10	C_1									

Column Dominates Row

Partial Ordering C_r

Consider any r satisfying $0 \leq r \leq 1$

Define

$W_r = \{w' \in S : w' = (1 - r)w^0 + rw \text{ for some } w \in S\}$

Interpretation

$1 - r$ degree of confidence in initial weighting w^0

Ellsburg (1961)

r "size" of resulting set for checking robustness

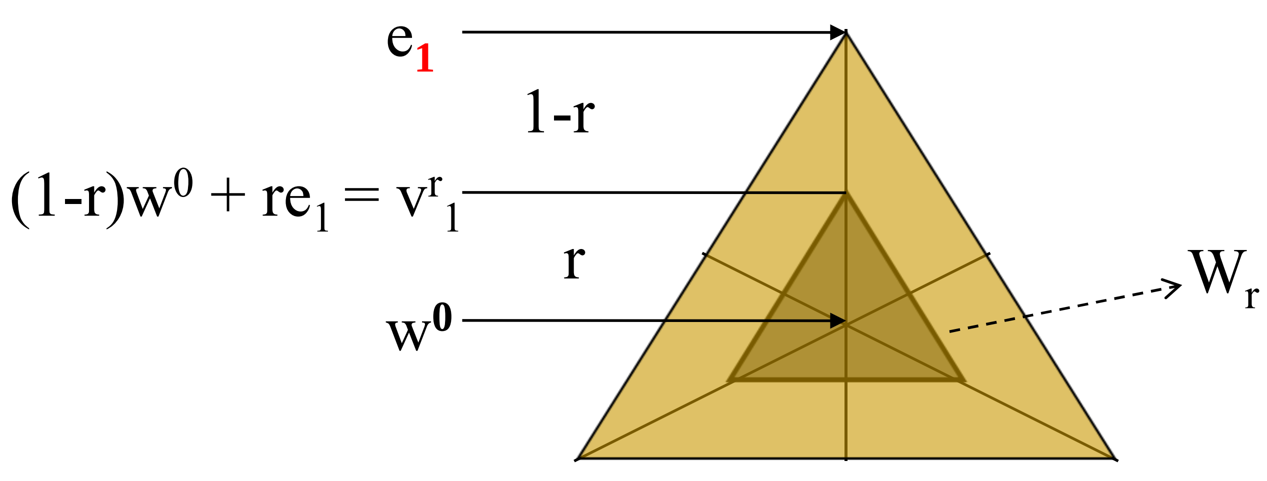
Ex

$W_1 = S$ lowest degree of confidence in w^0, largest set

$W_0 = \{w^0\}$ highest degree of confidence, smallest set

W_r = intermediate

Partial Ordering C_r



How should the set W_r look like?

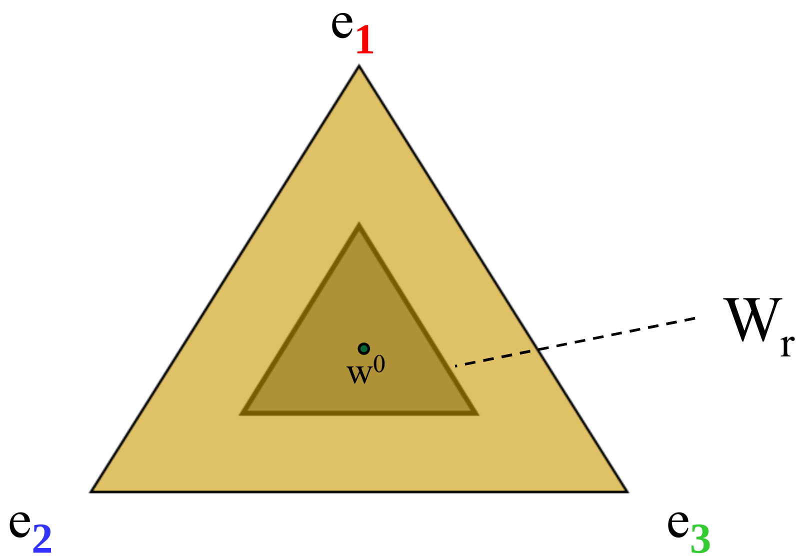
Partial Ordering C_r

Suppose W = W_r

Denote resulting ranking by C_r so that

$x\ C_r\ y$ if and only if $C(x;w^0) > C(y;w^0)$ and

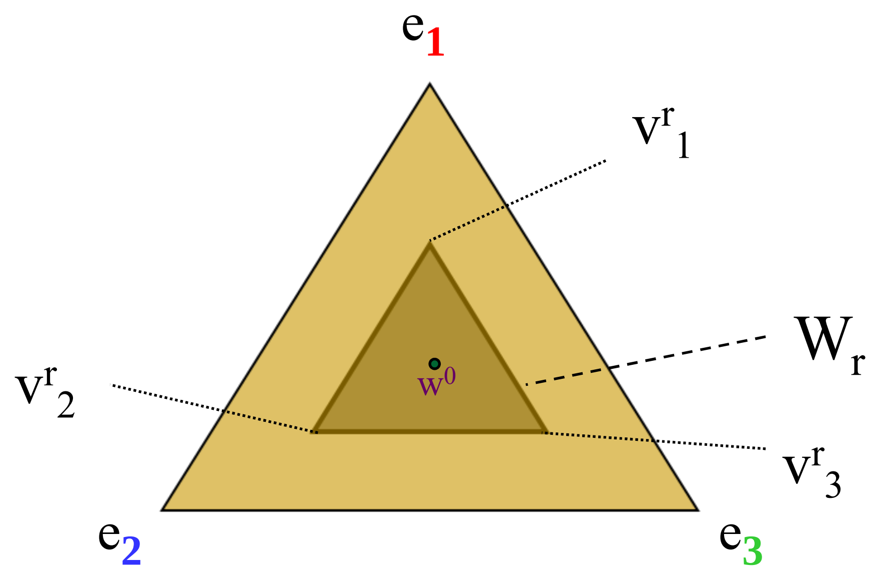
$C(x;w) \geq C(y;w)$ for all $w \in W_r$



Characterization of C_r

Denote

$v^r_d = (1- r)w^0 + re_d;$

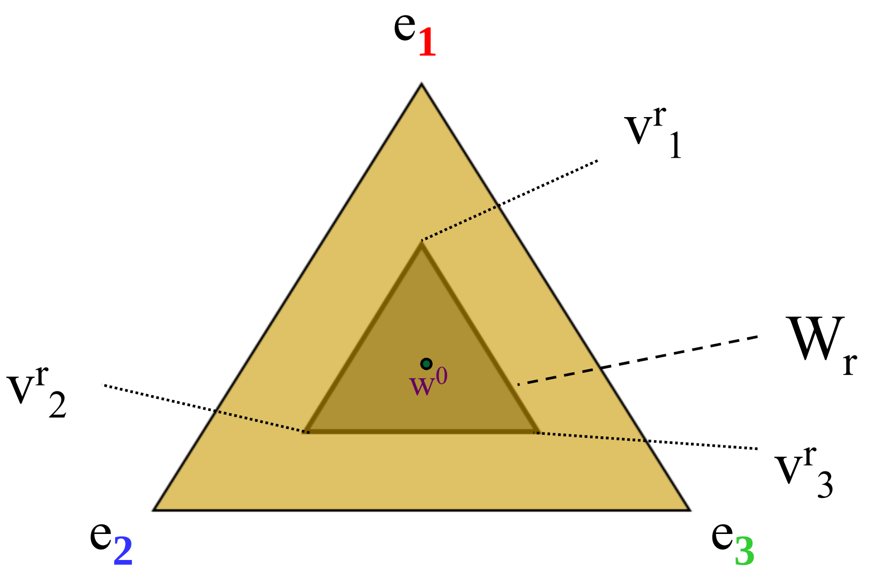


$x^r_d = v^r_d \cdot x$ Value of x at v^r_d

$x^r = (x^r_1, \ldots, x^r_D)$ Vector of these values

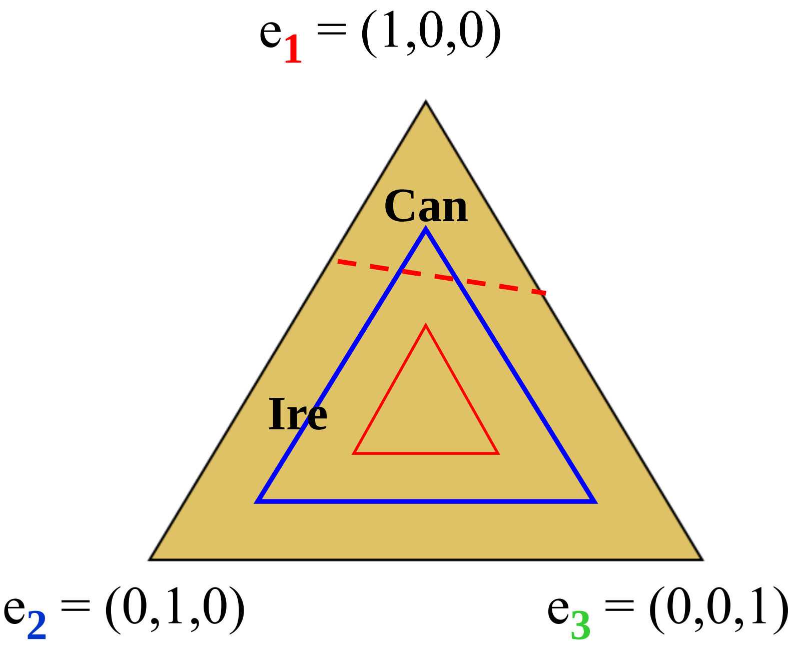
Characterization of C_r

Theorem 2: x C_r y if and only if $x^r \geq y^r$ and $x^r \neq y^r$



Partial Ordering C_r

Recall Canada/Ireland example



Robust?

– **No!**

Robust?

– **Yes.**

Recall Fully Robust Ranking C_1

Coun.		Nor	Ice	Aus	Ire	Swe	Can	Jap	USA	Swit	Neth
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2										
Aus	3										
Ire	4										
Swe	5			C_1							
Can	6		C_1								
Jap	7										
USA	8										
Swit	9		C_1								
Neth	10	C_1									

Column Dominates Row

C_r for r = 1/4

Coun.		Nor	Ice	Aus	Ire	Swe	Can	Jap	USA	Swit	Neth
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2										
Aus	3	C_r									
Ire	4	C_r									
Swe	5	C_r	C_r	C_r							
Can	6	C_r	C_r	C_r							
Jap	7	C_r	C_r								
USA	8	C_r	C_r		C_r						
Swit	9	C_r	C_r	C_r							
Neth	10	C_r	C_r	C_r	C_r	C_r					

Column Dominates Row

Measure of Robustness

Idea

How robust is a given comparison?

Denote

$D_0 = C(x,w^0) - C(y,w^0)$ difference in HDI's

$D_M = \max_{w \in S}\{C(y,w) - C(x,w)\}$ max dimensional departure

Define

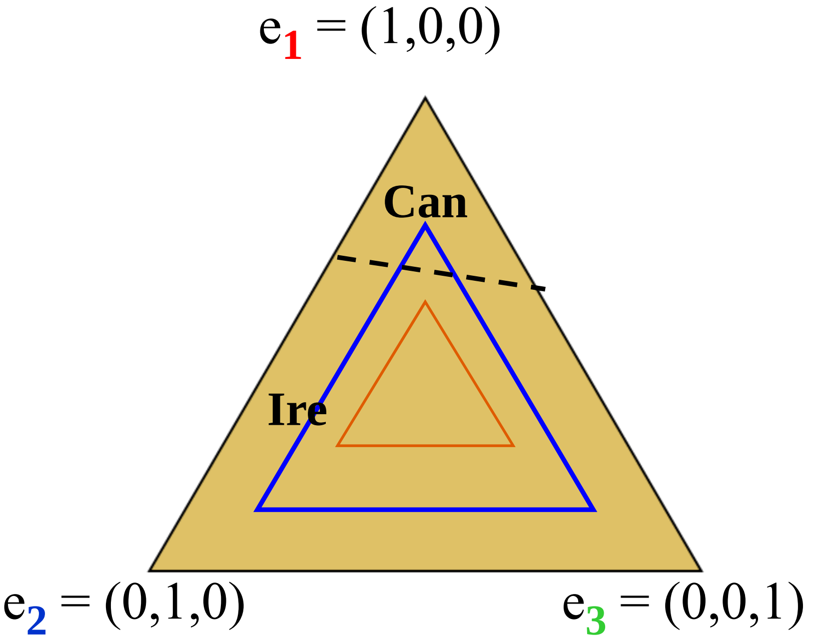
$r^* = D_0/(D_0 + D_M)$ Measure of robustness

Measure of Robustness

Theorem 3: Suppose that x C_0 y for x, y $\in$ X and let r* be the robustness level associated with this comparison. Then x C_r y holds for $0 < r \le r^*$ and does not for $r^* < r \le 1$.

Difficult to follow what is going on?

Measure of Robustness



Size of maximum possible simplex is robustness level r^*

Robustness Calculation

Coun.	**HDI**	x_1	x_2	x_3
Ire	0.956	0.882	0.990	0.995
Can	0.950	0.919	0.970	0.959
Difference	0.006	-0.037	0.02	0.036

$D_0 = 0.006$

$D_M = 0.037$

$r^* = 0.006/(0.006 + 0.037)$

$= 0.139$

Robustness Calculation

Coun.	**HDI**	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$
Aus	0.957	0.925	0.993	0.954
Swe	0.951	0.922	0.982	0.949
Difference	0.006	0.003	0.011	0.005

- $D_0 = 0.006$
- $D_M = - 0.003$
- $r^* = 1$

Note that both comparisons have same D_0

Yet very different robustness levels!

Measure of Robustness (%)

Coun.		Nor	Ice	Aus	Ire	Swe	Can	Jap	USA	Swit	Neth
	Rank	1	2	3	4	5	6	7	8	9	10
Nor	1										
Ice	2	20									
Aus	3	**35**	19								
Ire	4	**86**	14	4							
Swe	5	**53**	**94**	**100**	11						
Can	6	**61**	**100**	**60**	14	14					
Jap	7	**28**	**34**	23	9	7	2				
USA	8	**77**	**28**	17	**67**	5	3	1			
Swit	9	**49**	**100**	**41**	16	17	20	6	2		
Neth	10	**100**	**68**	**57**	**47**	**25**	13	4	7	1	

Column Dominates Row

Prevalence of Robust Comparisons

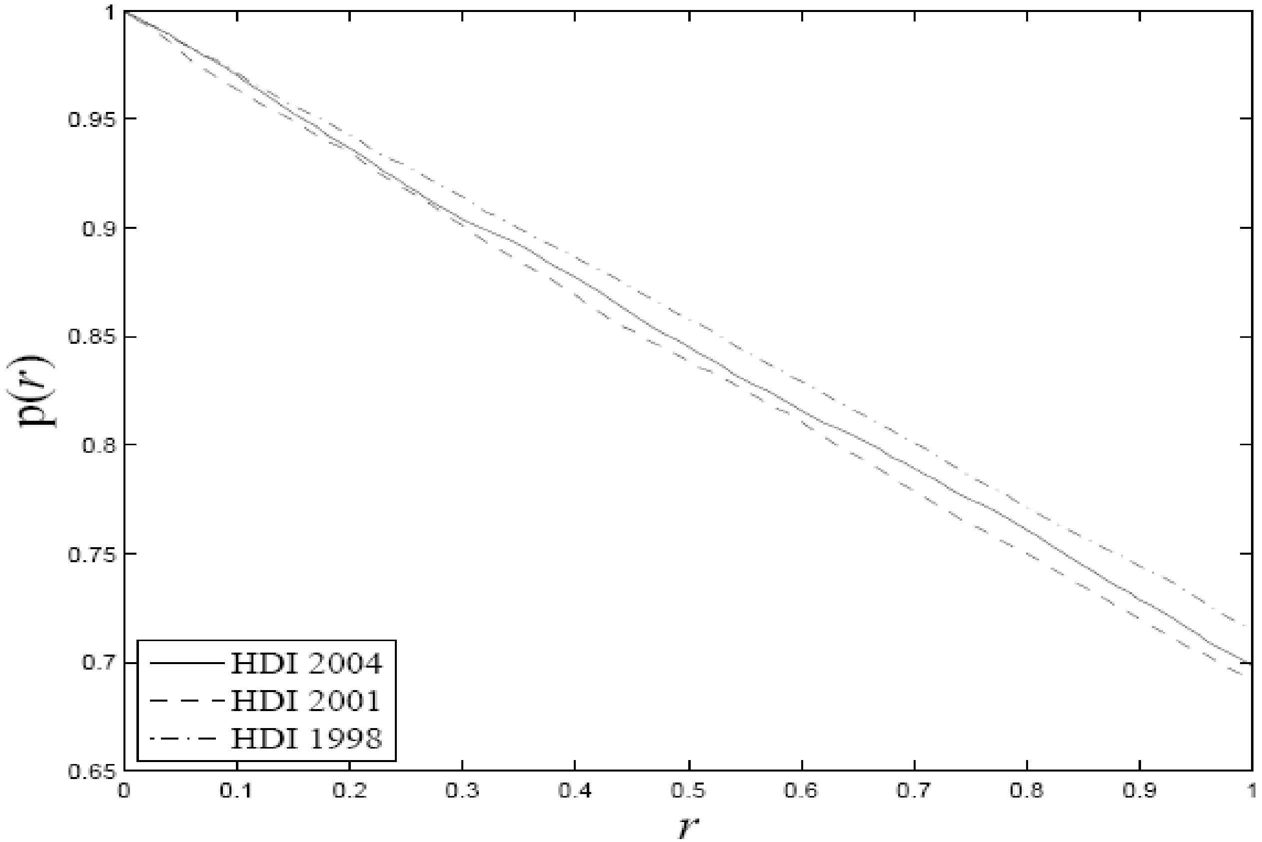
Consider a specific dataset

- n Countries
- Total Comparisons: $N = n(n-1)/2$

Given robustness level r

- Number of robust Comparisons: N_r
- Proportion of Robust Comparison: $p(r) = N_r/N$
- Prevalence function $p:[0,1] \rightarrow [0,1]$

Prevalence Functions



HDI: Human Development Index

69-72 percent comparisons are totally robust

Prevalence of Robustness and Association Among Dimensions

- What is the value of p(1) when dimensions are perfectly positively associated?
  - 100 percent
- What is the value of p(1) when any two dimensions are perfectly negatively associated?
  - Zero percent
- As multidimensional association among dimensions increases, p(1) goes up

Example

- Perfect positive association

	Dimension 1	Dimension 2	Dimension 3
Region 1	0.9	0.9	0.9
Region 2	0.8	0.7	0.6
Region 3	0.6	0.5	0.5

- Which comparisons are robust?
  - All of them are fully robust

Summary

- Initial Weighting System is arbitrary
- Introduced *Partial orderings* for robust comparisons
- Proposed a measure of robustness
- Prevalence function for various HDI dataset
- Prevalence of robustness and multidimensional association
- Can be extended to the case of general means

Thank you