OPHI
Oxford Poverty & Human Development Initiative
University of Oxford
www.ophi.org.uk

Summer School on Capability and Multidimensional Poverty

10-20 September, 2010

Amman, Jordan

Properties of Multidimensional Poverty Measures

Maria Emma Santos
(CONICET-UNS and OPHI)

Typical Dataset

- Where x_{ij} is the achievement of individual i of attribute or dimension j

Dimensions

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}$$

People

nxd

This Lecture…

- Focus of this lecture: The axiomatic structure considered as ‘desirable’ or ‘convenient’ for the measurement of inequality and poverty in the multidimensional context.

Type of Axioms in the Multidimensional Context

- Natural extensions of the unidimensional framework.

- Axioms specific to the multidimensional context (considering the potential relationships between the attributes or dimensions).

Axioms in Multidimensional Poverty Measurement

Axioms in Multidimensional Poverty Measurement

- Borrows from:
  - Axioms of Unidimensional Poverty
  - Axioms of Multidimensional Inequality

Notation

- Let matrix X=[xij], of size nxd the multidimensional distribution of d attributes among n individuals, with non-negative elements.

- Let Ω be the set of all nxd matrices.

- Let vector $z \in R^{d}_{++}$ be the cutoff vector containing the poverty line of each dimension.

Notation-Steps to measurement

- **Identification:** Who is multidimensionally poor? We need an 'identification function', a criterion that decides who is considered multidimensionally poor.

$$\rho : R^d_+ \times R^d_{++} \rightarrow \{0,1\}$$

$$\rho(x_i, z) = 1 \quad \text{if} \quad i \quad \text{multi.poor}$$

$$\rho(x_i, z) = 0 \quad \text{if} \quad i \quad \text{not} \quad \text{multi.poor}$$

- Applying the identification function ρ, we get the set of the multidimensionally poor, name it Z (<u>different from the vector of poverty lines z</u>)

Notation-Steps to measurement

- **Aggregation:** Takes the identification criterion as given and associates with the matrix Y and the cutoff vector z and overall level P(Y,z) of multidimensional poverty. $P : \Omega \times R^{d}_{++}$

Typical Dataset

- Where x_{ij} is the achievement of individual i of attribute or dimension j .
- zj is the poverty line of attribute or dimension j.

Dimensions

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}$$

People

$$z = (z_1, z_2, \ldots\ldots, z_d)$$

Typical Dataset

- Example: Suppose income, years of education and self-rated health. Suppose that the poverty lines are 5 for income, 6 for years of education and 4 for health.

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Anonymity (Symmetry)

X is obtained from Y by a ***permutation*** of attributes between individuals if X=PY, where P is a nxn permutation matrix. Matrix X is matrix Y with rows interchanged.

$$X = PY = \begin{bmatrix} 010 \\ 100 \\ 001 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 4 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$$

SYMMETRY (Anonymity): If X is obtained from *y* by a ***permutation***, then P(X;z)=P(Y;z).

Replication Invariance

X is obtained from Y by a k-***replication of people*** if it is constructed by replicating each *i*th individual's attributes distribution a number of times (ie. a replication of each row in matrix).

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

REPLICATION INVARIANCE (Population Principle):

If X is obtained from Y by a ***replication***, then P(X;z)=P(Y;z).

Scale Invariance

$(X';z')$ is obtained from $(X;z)$ by a ***proportional change*** if $(X';z') = (\alpha X;\alpha z)$ for $\alpha > 0$

$$X = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \quad z' = \begin{bmatrix} 2z_1 & 2z_2 & 2z_3 \end{bmatrix}$$

SCALE INVARIANCE (Zero-Degree Homogeneity): If $(X';z')$ is obtained from $(X;z)$ by a ***proportional change***, then P(X';z')=P(X;z).

Focus: Two types

- Focus on those identified as ‘multidimensionally poor’ (we are not interested in those not multidimensionally poor)

- Focus on the attributes in which they are deprived (we are not interested in dimensions or attributes in which they are not deprived).

Focus on the ‘multidimensionally poor’

X is obtained from Y by a ***simple increment among the non-poor*** if:

$$x_{i'j'} > y_{i'j'} \quad for \quad (i,j) = (i',j') \quad if \quad i' \notin Z$$

$$x_{ij} = y_{ij} \quad for \quad (i,j) \neq (i',j')$$

POVERTY FOCUS: If X is obtained from Y by ***a simple increment among the non-poor***, then P(X;z)=P(Y;z).

Example of Poverty Focus

- Suppose person 3 is not considered multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Focus on deprived attributes

X is obtained from Y by a ***simple increment among the non-deprived*** if:

$x_{i'j'} > y_{i'j'}$ for $(i, j) = (i', j')$ if $y_{i'j'} \geq z_{j'}$

$x_{ij} = y_{ij}$ if $(i, j) \neq (i', j')$

DEPRIVATION FOCUS: If X is obtained from Y by ***a simple increment among the non-deprived***, then P(X;z)=P(Y;z).

Example of Deprivation Focus

- Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increase in health (dimension in which he/she is not deprived)?

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Focus

- One type of focus axiom does not necessarily imply the other one.

- Example: Suppose that a unidimensional approach was used to measure multidimensional poverty, assigning a price to each attribute, adding them up and comparing them with an income z. Then an increment in a non-deprived attribute could compensate deprivation in another and reduce poverty. In this case, poverty focus would be satisfied, but deprivation focus would not.

Continuity

P(X;z) is continuous if a small change in any attribute does not result in an abrupt change in the poverty index P(X;z)

- *CONTINUITY:* For any sequence X^k, if X^k converges to X, then $P(X^k;z)$ converges to $P(X;z)$

Monotonicity: Two types

- Becoming less deprived in a specific attribute (within dimension): Monotonicity

- Stop being deprived in one dimension (across dimensions): Dimensional Monotonicity

Monotonicity (within dimension)

X is obtained from Y by a ***deprived increment among the poor*** if for some *i*:

$x_{i'j'} > y_{i'j'}$ for $(i, j) = (i', j')$ where

$y_{i'j'} < z_{j'}$ and $i' \in Z$

while $x_{ij} = y_{ij}$ for all $(i, j) \neq (i', j')$

MONOTONICITY: If X is obtained from Y by a ***deprived increment*** among the poor, then $P(X,z)<P(Y,z)$.

Example of Monotonicity (within dimension)

- Suppose person 1 is considered multidimensionally poor, and experiences an improvement in his/her health.

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Dimensional Monotonicity (across dimensions)

X is obtained from Y by a ***dimensional increment among the poor*** if for some *i*:

$$x_{i'j'} \geq z_{j'} > y_{i'j'} \quad \text{for} \quad (i,j)=(i',j') \quad \text{where} \quad i' \in Z$$

$$\text{while} \quad x_{ij} = y_{ij} \quad \text{for} \quad \text{all} \quad (i,j) \neq (i',j')$$

DIMENSIONAL MONOTONICITY: If X is obtained from y by a ***dimensional increment among the poor***, then P(X,z)<P(Y,z).

Example of Dimensional Monotonicity (across dimension)

- Suppose person 2 is considered multidimensionally poor, and experiences an increment in his/her education such that now is no longer deprived in this dimension.

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 6 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Extensions of the Pigou-Dalton Transfer Principle (I)

X is obtained from Y by **Uniform Pigou Dalton Transfer among the poor** (UPD) (an averaging of achievements between two poor people) if Y=TX, where $T=\lambda E+(1-\lambda)P$ (with $0<\lambda<1$), with $t_{ii}=1$ for every non-poor person *i* in Y, E is an identity matrix and P is a permutation matrix that interchanges two rows.

$$T=\frac{1}{2}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&1\end{bmatrix}$$

$$X=TY=\begin{bmatrix}0.5&0.5&0\\0.5&0.5&0\\0&0&1\end{bmatrix}\begin{bmatrix}4&4&2\\3&5&4\\8&6&3\end{bmatrix}=\begin{bmatrix}3.5&4.5&3\\3.5&4.5&3\\8&6&3\end{bmatrix}$$

The distribution of the attributes has been smoothed between individuals 1 and 2.

‘d progressive transfers’

Extensions of the Pigou-Dalton Transfer Principle (I)

- *TRANSFER UNDER UPD :* If X is obtained from *Y* by a ***Uniform Pigou-Dalton Transfer*** among the poor (an averaging of achievements between two people), then $P(X;z) \leq P(Y;z)$.

Extensions of the Pigou-Dalton Transfer Principle (II-Kolm, 1977)

X is obtained from Y by a ***Uniform Majorization among the poor*** (an averaging of achievements among the poor) if X=BY, where B is a nxn bistochastic matrix but not a permutation matrix, and $b_{ii}=1$ for every non-poor person *i* in Y.

$$X = BY = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 & 3 \\ 3.4 & 4.5 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

The distribution of the attributes has been smoothed among the two poor individuals. In this particular case it is the same as the previous UPD because there are only two poor people. But this does not need to be always the case.

Example of a UM that is not a UPD

Suppose there are three poor individuals and one non-poor. The following transformation is a UM which does not satisfy the definition of the UPD transformation as it affects more than two people.

$$X = BY = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 5 & 3 & 3 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 4 \\ 4 & 4 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

Extensions of the Pigou Dalton Transfer Principle (II)

- *TRANSFER UNDER UM :* If X is obtained from *Y* a ***uniform majorization among the poor*** (an averaging of achievements among the poor), then P(X;z)≤P(Y;z).

UPD and UM

- Any T matrix is a B matrix, but the converse is not true. Then UDP implies UM but the converse is not true.

- Only when n=2 or k=1, UPD is equivalent to UM.

Axiom Specific to the Multidimensional Case

- Considers the ‘correlation’ between attributes when marginal distributions are unchanged (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

- Intrinsic to the multivariate case.

Axiom Specific to the Multidimensional Case

- Example: A switch in dimension 3 between person 1 and 2

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \qquad Y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix}$$

Ways to call this transfer:

- From X to Y: association increasing rearrangement, (Alkire & Foster, 2007).
- From Y to X: association decreasing rearrangement (Boland & Proschan, 1988); correlation-increasing transfer (Tsui, 1999), correlation increasing switch (Bourguignon & Chakravarty, 2003).

Axiom Specific to the Multidimensional Case

X is obtained from Y by an ***association decreasing rearrangement among the poor*** if for two persons *i* and *i':*

i) *i* and i' are poor in Y

ii) Either they have the same amount of attribute in X than in Y: $(x_{ij},x_{i'j})=(y_{ij},y_{i'j})$

iii) Or they have switched the amount of the attribute: $(x_{ij},x_{i'j})=(y_{i'j},y_{ij})$

iii) The amount of attributes of the other people $i''\neq i,i'$ remain unchanged: $x_{i''j}=y_{i''j}$

iv) Vectors of *i* and *i''* are comparable by vector dominance in Y but not in X

Question…

- How do you think poverty should change under an association decreasing rearrangement?

Association Decreasing Rearrangement

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \qquad Y = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix}$$

- If you think that a good health can ***substitute*** for a bad income or a bad education, then poverty should ***decrease***.
- If you think that a good health is ***necessary (complementary)*** to achieve a good income and a good education, then poverty should ***increase***.
- If you think that health is not necessary to achieve a good income and a good education, and can not either substitute for any of these, (ie: you think they are ***independent***), then poverty should ***not change***.

Axiom Specific to the Multidimensional Case

- *Weak Rearrangement :* If X is obtained from *Y* by an ***association decreasing rearrangement among the poor***, $P(X;z) \leq P(Y;z)$.

Axiom Specific to the Multidimensional Case

- What is the assumption behind the axiom?
  - Assumption: Attributes are ***independent*** (if =) or ***substitutes*** (if <) (compensating inequalities).
  - Could be ***complements***, then axiom should go in the other way (>). (Bourguignon & Chakravarty, 2003).

ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

$U(x_1,x_2)$ such that:

- marginal utility for each attribute is non-decreasing: $\frac{\partial U}{\partial x_1} \geq 0, \frac{\partial U}{\partial x_2} \geq 0$

- If cross derivative is non-positive, attributes are substitutes: $\frac{\partial^2 U}{\partial x_1 x_2} \leq 0$

(Utility increases less with an increase in attribute 2 for persons with larger quantities of attribute 1).

- If cross derivative is non-negative, attributes are complements $\frac{\partial^2 U}{\partial x_1 x_2} \geq 0$

(Utility increases more with an increase in attribute 2 for persons with larger quantities of attribute 1).

ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

- The cross derivative condition has to hold for the ‘least concave utility function’, so that the condition holds under monotonic transformations of the utility function. (Kannai, 1980).

Population Subgroup Consistency

If $P(X_1;z)>P(Y_1;z)$ and $P(X_2;z)=P(Y_2;z)$, and $n(X_1)=n(Y_1)$, $n(X_2)=n(Y_2)$, then $P(X_1,Y_1)>P(X_2,Y_2;z)$.

Decomposability

- *Population Subgroup Decomposability:*

  A poverty measure P is decomposable if $P(X,Y;z)=(n(X)/n)P(X)+(n(Y)/n)P(Y)$

- Extending this axiom to subgroups of dimensions in a strict way is not straightforward. It is called break-down by dimension, because it is dependent on the identification criterion of the multidimensionally poor. See Alkire & Foster (2007)

Example of Population Subgroups

Income Education Health

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

	Income	Education	Health
Person 1	4	4	2
Person 2	3	5	4
Person 3	8	6	3

Example of Dimension Subgroups

Income Education Health

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \begin{matrix} \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{matrix}$$