Summer School on Capability and Multidimensional Poverty
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Properties of Multidimensional Poverty Measures

Maria Emma Santos (OPHI)
Typical Dataset

- Where $x_{ij}$ is the achievement of individual $i$ of attribute or dimension $j$

$$X = \begin{bmatrix}
X_{11} & \cdots & X_{1d} \\
X_{21} & \cdots & X_{2d} \\
\vdots & & \vdots \\
X_{n1} & \cdots & X_{nd}
\end{bmatrix}$$

$n \times d$
This Lecture…

• Focus of this lecture: The axiomatic structure considered as ‘desirable’ or ‘convenient’ for the measurement of inequality and poverty in the multidimensional context.
Type of Axioms in the Multidimensional Context

• Natural extensions of the unidimensional framework.

• Axioms specific to the multidimensional context (considering the potential relationships between the attributes or dimensions).
Axioms in Multidimensional Poverty Measurement
Axioms in Multidimensional Poverty Measurement

• Borrows from:
  – Axioms of Unidimensional Poverty
  – Axioms of Multidimensional Inequality
Notation

• Let matrix $X = [x_{ij}]$, of size $n \times d$ the multidimensional distribution of $d$ attributes among $n$ individuals, with non-negative elements.

• Let $\Omega$ be the set of all $n \times d$ matrices.

• Let vector $z \in \mathbb{R}^d_{++}$ be the cutoff vector containing the poverty line of each dimension.
Notation-Steps to measurement

• **Identification:** Who is multidimensionally poor? We need an ‘identification function’, a criterion that decides who is considered multidimensionally poor.

\[
\rho : \mathbb{R}_+^d \times \mathbb{R}^{++,d} \rightarrow \{0, 1\}
\]

\[
\rho(x_i, z) = 1 \quad \text{if} \quad i \quad \text{multi. poor}
\]

\[
\rho(x_i, z) = 0 \quad \text{if} \quad i \quad \text{not multi. poor}
\]

• Applying the identification function \(\rho\), we get the set of the multidimensionally poor, name it \(Z\) (different from the vector of poverty lines \(z\))
Notation-Steps to measurement

• **Aggregation:** Takes the identification criterion as given and associates with the matrix $Y$ and the cutoff vector $z$ and overall level $P(Y,z)$ of multidimensional poverty. $P : \Omega \times \mathbb{R}^d_{++}$
Typical Dataset

• Where \( x_{ij} \) is the achievement of individual \( i \) of attribute or dimension \( j \).
• \( z_j \) is the poverty line of attribute or dimension \( j \).

\[
X = \begin{bmatrix}
X_{11} & \ldots & X_{1d} \\
X_{21} & \ldots & X_{2d} \\
\vdots & \ddots & \vdots \\
X_{n1} & \ldots & X_{nd}
\end{bmatrix}
\]

\[
z = (z_1, z_2, \ldots, z_d)
\]
Typical Dataset

• Example: Suppose income, years of education and self-rated health. Suppose that the poverty lines are 5 for income, 6 for years of education and 4 for health.

\[
X = \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix} \quad g^0 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
z = (5, 6, 4)
\]
Anonymity (Symmetry)

X is obtained from Y by a permutation of attributes between individuals if X=PY, where P is a nxn permutation matrix. Matrix X is matrix Y with rows interchanged.

\[
X = PY = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3 \\
\end{bmatrix}
= \begin{bmatrix}
3 & 5 & 4 \\
4 & 4 & 2 \\
8 & 6 & 3 \\
\end{bmatrix}
\]

Symmetry (Anonymity): If X is obtained from y by a permutation, then P(X;z)=P(Y;z).
Replication Invariance

X is obtained from Y by a k-\textit{replication of people} if it is constructed by replicating each $i$th individual’s attributes distribution a number of times (ie. a replication of each row in matrix).

\[
\begin{align*}
Y &= \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3 \\
\end{bmatrix} & X &= \begin{bmatrix}
4 & 4 & 2 \\
4 & 4 & 2 \\
3 & 5 & 4 \\
3 & 5 & 4 \\
8 & 6 & 3 \\
8 & 6 & 3 \\
\end{bmatrix}
\end{align*}
\]

\textit{Replication Invariance (Population Principle)}: If X is obtained from Y by a \textit{replication}, then $P(X;z)=P(Y;z)$. 

Scale Invariance

$(X';z')$ is obtained from $(X;z)$ by a proportional change if $(X';z') = (\alpha X; \alpha z)$ for $\alpha > 0$

$$X = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \quad \quad z' = [2z_1, 2z_2, 2z_3]$$

**Scale Invariance (Zero-Degree Homogeneity):** If $(X';z')$ is obtained from $(X;z)$ by a proportional change, then $P(X';z')=P(X;z)$. 
Focus: Two types

- Focus on those identified as ‘multidimensionally poor’ (we are not interested in those not multidimensionally poor)

- Focus on the attributes in which they are deprived (we are not interested in dimensions or attributes in which they are not deprived).
Focus on the ‘multidimensionally poor’

X is obtained from Y by a *simple increment among the non-poor* if:

\[ x_{i',j} > y_{i',j} \quad \text{for} \quad (i, j) = (i', j') \quad \text{if} \quad i' \in Z \]

\[ x_{i,j} = y_{i,j} \quad \text{for} \quad (i, j) \neq (i', j) \]

**Poverty Focus:** If X is obtained from Y by *a simple increment among the non-poor*, then \( P(X;z) = P(Y;z) \).
Example of Poverty Focus

• Suppose person 3 is not considered multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?

\[ X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}, \quad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z = (5, 6, 4) \]
Focus on deprived attributes

X is obtained from Y by a simple increment among the non-deprived if:

\[ x_{i'j'} > y_{i'j'} \quad \text{for} \quad (i, j) = (i', j') \quad \text{if} \quad y_{i'j'} \geq z_{j'} \]

\[ x_{ij} = y_{ij} \quad \text{if} \quad (i, j) \neq (i', j') \]

**Poverty Focus:** If X is obtained from Y by a simple increment among the non-deprived, then \( P(X;z) = P(Y;z) \).
Example of Deprivation Focus

• Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increase in health (dimension in which he/she is not deprived)

\[
X = \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3 \\
\end{bmatrix} \quad g^0 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \\
z = (5, 6, 4)
\]
Focus

• One type of focus axiom does not necessarily imply the other one.

• Example: If a unidimensional approach was used to measure multidimensional poverty, assigning a price to each attribute, adding them up and comparing them with an income $z$, then an increment in a non-deprived attribute could compensate deprivation in another and reduce poverty, then poverty focus would be satisfied, but deprivation focus would not.
Continuity

$P(X;z)$ is continuous if a small change in any attribute does not result in an abrupt change in the poverty index $P(X;z)$

- **CONTINUITY:** For any sequence $X^k$, if $X^k$ converges to $X$, then $P(X^k;z)$ converges to $P(X;z)$
Monotonicity: Two types

• Becoming less deprived in a specific attribute (within dimension): Monotonicity

• Stop being deprived in one dimension (across dimensions): Dimensional Monotonicity
Monotonicity (within dimension)

X is obtained from Y by a deprived increment among the poor if for some i:
\[ x_{i'j'} > y_{i'j'} \quad \text{for} \quad (i, j) = (i', j') \quad \text{where} \]
\[ y_{i'j'} < z_j \quad \text{and} \quad i' \in Z \]
while \[ x_{ij} = y_{ij} \quad \text{for} \quad \text{all} \quad (i, j) \neq (i', j') \]

**Monotonicity:** If X is obtained from Y by a deprived increment among the poor, then \[ P(X, z) < P(Y, z). \]
Example of Monotonicity (within dimension)

- Suppose person 1 is considered multidimensionally poor, and experiences an improvement in his/her health.

\[
Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \quad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}
\]

\[z = (5, 6, 4)\]
Dimensional Monotonicity (across dimensions)

X is obtained from Y by a *dimensional increment among the poor* if for some $i$:

$$x_{i'j'} \geq z_{j'} > y_{i'j'} \quad \text{for} \quad (i, j) = (i', j') \quad \text{where} \quad i' \in Z$$

while $x_{ij} = y_{ij} \quad \text{for all} \quad (i, j) \neq (i', j')$

**Dimensional Monotonicity**: If X is obtained from y by a *dimensional increment among the poor*, then $P(X,z) < P(Y,z)$. 
Example of Dimensional Monotonicity (across dimension)

• Suppose person 2 is considered multidimensionally poor, and experiences an increment in his/her education such that now is no longer deprived in this dimension.

\[
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix}
\quad Y =
\begin{bmatrix}
4 & 4 & 2 \\
3 & 6 & 4 \\
8 & 6 & 3
\end{bmatrix}
\quad X =
\]

\[z = (5, 6, 4)\]
Extensions of the Pigou-Dalton Transfer Principle (I)

X is obtained from Y by **Uniform Pigou Dalton Transfer among the poor** (UPD) (an averaging of achievements between two poor people) if Y=TX, where $T=\lambda E+ (1-\lambda)P$ (with $0<\lambda<1$), with $t_{ii}=1$ for every non-poor person $i$ in Y, $E$ is an identity matrix and $P$ is a permutation matrix.

$$
T = \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1/2 & 1/2 & 0 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

The distribution of the attributes has been smoothed between individuals 1 and 2.

```
X = TY = \begin{bmatrix}
0.5 & 0.5 & 0 \\
0.5 & 0.5 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix} = \begin{bmatrix}
3.5 & 4.5 & 3 \\
3.5 & 4.5 & 3 \\
8 & 6 & 3
\end{bmatrix}
```

‘d progressive transfers’
Extensions of the Pigou-Dalton Transfer Principle (I)

- **Transfer Under UPD**: If $X$ is obtained from $Y$ by a *Uniform Pigou-Dalton Transfer* among the poor (an averaging of achievements between two people), then $P(X;z) \leq P(Y;z)$. 
Extensions of the Pigou-Dalton Transfer Principle (II-Kolm, 1977)

X is obtained from Y by multiplying it by a
Uniform Majorization among the poor (an averaging of achievements among the poor) if X=BY, where B is a nxn bistochastic matrix but not a permutation matrix, and b_{ii}=1 for every non-poor person i in Y.

\[
X = BY = \begin{bmatrix}
0.5 & 0.5 \\
0.5 & 0.5 \\
1
\end{bmatrix}
\begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3
\end{bmatrix}
= \begin{bmatrix}
3.5 & 4.5 & 3 \\
3.4 & 4.5 & 3 \\
8 & 6 & 3
\end{bmatrix}
\]

The distribution of the attributes has been smoothed among all individuals.
Extensions of the Pigou Dalton Transfer Principle (II)

- **TRANSFER UNDER UM**: If X is obtained from Y a uniform majorization among the poor (an averaging of achievements among the poor), then $P(X;z) \leq P(Y;z)$. 
UPD and UM

• Any $T$ matrix is a $B$ matrix, but the converse is not true. Then UDP implies UM but the converse is not true.

• Only when $n=2$ or $k=1$, UPD is equivalent to UM.
Axiom Specific to the Multidimensional Case

• Considers the ‘correlation’ between attributes when marginal distributions are unchanged (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

• Intrinsic to the multivariate case.
Axiom Specific to the Multidimensional Case

- Example: A switch in dimension 3 between person 2 and 3

\[
Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix}
\]

Ways to call this transfer:

- From Y to X: association increasing rearrangement, (Alkire & Foster, 2007).
- From X to Y: association decreasing rearrangement (Boland & Proschan, 1988); correlation-increasing transfer (Tsui, 1999), correlation increasing switch (Bourguignon & Chakravarty, 2003).
Axiom Specific to the Multidimensional Case

X is obtained from Y by an association decreasing rearrangement among the poor if for two persons i and i’:

i) i and i‘ are poor in Y

ii) Either they have the same amount of attribute in X than in Y: \((x_{ij},x_{i'j})=(y_{ij},y_{i'j})\)

iii) Or they have switched the amount of the attribute: \((x_{ij},x_{i'j})=(y_{i'j},y_{ij})\)

iii) The amount of attributes of the other people \(i''\neq i,i'\) remain unchanged: \(x_{i''j}=y_{i''j}\)

iv) Vectors of i and i’’ are comparable by vector dominance in y but not in x
Question…

• How do you think poverty should change under an association decreasing rearrangement?
Association Decreasing Rearrangement

\[ Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \quad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix} \]

- If you think that a good health can substitute for a bad income or a bad education, then poverty should decrease.
- If you think that a good health is necessary (complementary) to achieve a good income and a good education, then poverty should increase.
- If you think that health is not necessary to achieve a good income and a good education, and can not either substitute for any of these, (ie: you think they are independent), then poverty should not change.
Axiom Specific to the Multidimensional Case

- **Weak Rearrangement**: If $X$ is obtained from $Y$ by an association decreasing rearrangement among the poor, $P(X;z) \leq P(Y;z)$. 

\[ P(X;z) \leq P(Y;z). \]
Axiom Specific to the Multidimensional Case

• What is the assumption behind the axiom?
  – Assumption: Attributes are independent (if =) or substitutes (if <) (compensating inequalities).
  – Could be *complements*, then axiom should go in the other way (>). (Bourguignon & Chakravarty, 2003).
ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

\[ U(x_1,x_2) \] such that:

- marginal utility for each attribute is non-decreasing: 
  \[ \frac{\partial U}{\partial x_1} \geq 0, \frac{\partial U}{\partial x_2} \geq 0 \]

- If cross derivative is non-positive, attributes are substitutes: 
  \[ \frac{\partial^2 U}{\partial x_1 \partial x_2} \leq 0 \]

  (Utility increases less with an increase in attribute 2 for persons with larger quantities of attribute 1).

- If cross derivative is non-negative, attributes are complements 
  \[ \frac{\partial^2 U}{\partial x_1 \partial x_2} \geq 0 \]

  (Utility increases more with an increase in attribute 2 for persons with larger quantities of attribute 1).
ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

- The cross derivative condition has to hold for the ‘least concave utility function’, so that the condition holds under monotonic transformations of the utility function. (Kannai, 1980).
Subgroup Consistency

• **Population Subgroup Consistency:**

  If \( P(X_1;z) > P(Y_1;z) \) and \( P(X_2;z) = P(Y_2;z) \), and
  \( n(X_1) = n(Y_1) \), \( n(X_2) = n(Y_2) \), then \( P(X_1,Y_1) > P(X_2,Y_2;z) \).

Possible extension...

• **Dimension Subgroup Consistency:**

  If \( P(X_1;z) > P(Y_1;z) \) and \( P(X_2;z) = P(Y_2;z) \), and
  \( d(X_1) = d(Y_1) \), \( d(X_2) = d(Y_2) \), then \( P(X_1,Y_1) > P(X_2,Y_2;z) \).
Decomposability

• POPULATION SUBGROUP DECOMPOSABILITY:

  A poverty measure $P$ is decomposable if
  
  $$P(X,Y;z) = (n(X)/n)P(X) + (n(Y)/n)P(Y)$$

• Extending this axiom to subgroups of dimensions in a strict way is not straightforward. It is called breakdown by dimension, because it is dependent on the identification criterion of the multidimensionally poor.

  See Alkire & Foster (2007)
Example of Population Subgroups

<table>
<thead>
<tr>
<th>Income</th>
<th>Education</th>
<th>Health</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

$X = \begin{bmatrix}
4 & 4 & 2 \\
3 & 5 & 4 \\
8 & 6 & 3 \\
\end{bmatrix}$

Person 1
Person 2
Person 3
Example of Dimension Subgroups

<table>
<thead>
<tr>
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<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ X = \begin{bmatrix} 4 & 4 \\ 3 & 5 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \]

Person 1
Person 2
Person 3