OPHI

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Properties of Multidimensional Poverty Measures

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Typical Dataset

- Where x_{ij} is the achievement of individual i of attribute or dimension j

Dimensions

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}$$

People

nxd

This Lecture…

- Focus of this lecture: The axiomatic structure considered as ‘desirable’ or ‘convenient’ for the measurement of inequality and poverty in the multidimensional context.

Type of Axioms in the Multidimensional Context

- Natural extensions of the unidimensional framework.

- Axioms specific to the multidimensional context (considering the potential relationships between the attributes or dimensions).

Axioms in Multidimensional Poverty Measurement

Axioms in Multidimensional Poverty Measurement

- Borrows from:
  - Axioms of Unidimensional Poverty
  - Axioms of Multidimensional Inequality

Notation

- Let matrix X=[xij], of size nxd the multidimensional distribution of d attributes among n individuals, with non-negative elements.
- Let Ω be the set of all nxd matrices.
- Let vector $z \in R^{d}_{++}$ be the cutoff vector containing the poverty line of each dimension.

Notation-Steps to measurement

- **Identification:** Who is multidimensionally poor? We need an 'identification function', a criterion that decides who is considered multidimensionally poor.

$$\rho : R_{+}^{d} \times R_{++}^{d} \rightarrow \{0,1\}$$

$$\rho(x_i, z) = 1 \quad \text{if} \quad i \quad \text{multi. poor}$$

$$\rho(x_i, z) = 0 \quad \text{if} \quad i \quad \text{not} \quad \text{multi.poor}$$

- Applying the identification function ρ, we get the set of the multidimensionally poor, name it Z (<u>different from the vector of poverty lines z</u>)

Notation-Steps to measurement

- **Aggregation:** Takes the identification criterion as given and associates with the matrix Y and the cutoff vector z and overall level P(Y,z) of multidimensional poverty. $P : \Omega \times R^{d}_{++}$

Typical Dataset

Dimensions

- Where x_{ij} is the achievement of individual i of attribute or dimension j .
- zj is the poverty line of attribute or dimension j.

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \cdots & & \\ & & \cdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}$$

People

$$z = (z_1, z_2, \ldots\ldots, z_d)$$

Typical Dataset

- Example: Suppose income, years of education and self-rated health. Suppose that the poverty lines are 5 for income, 6 for years of education and 4 for health.

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Anonymity (Symmetry)

X is obtained from Y by a ***permutation*** of attributes between individuals if X=PY, where P is a nxn permutation matrix. Matrix X is matrix Y with rows interchanged.

$$X = PY = \begin{bmatrix} 010 \\ 100 \\ 001 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 4 \\ 4 & 4 & 2 \\ 8 & 6 & 3 \end{bmatrix}$$

SYMMETRY (Anonymity): If X is obtained from *y* by a ***permutation***, then P(X;z)=P(Y;z).

Replication Invariance

X is obtained from Y by a k-***replication of people*** if it is constructed by replicating each *i*th individual's attributes distribution a number of times (ie. a replication of each row in matrix).

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 3 & 5 & 4 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

REPLICATION INVARIANCE (Population Principle):

If X is obtained from Y by a ***replication***, then P(X;z)=P(Y;z).

Scale Invariance

$(X';z')$ is obtained from $(X;z)$ by a ***proportional change*** if $(X';z')=(\alpha X;\alpha z)$ for $\alpha > 0$

$$X = \begin{bmatrix} 2(4) & 2(4) & 2(2) \\ 2(3) & 2(5) & 2(4) \\ 2(8) & 2(6) & 2(3) \end{bmatrix} \quad z' = \begin{bmatrix} 2z_1 & 2z_2 & 2z_3 \end{bmatrix}$$

SCALE INVARIANCE (Zero-Degree Homogeneity): If

$(X';z')$ is obtained from $(X;z)$ by a ***proportional change***, then P(X‘;z')=P(X;z).

Focus: Two types

- Focus on those identified as 'multidimensionally poor' (we are not interested in those not multidimensionally poor)

- Focus on the attributes in which they are deprived (we are not interested in dimensions or attributes in which they are not deprived).

Focus on the ‘multidimensionally poor’

X is obtained from Y by a ***simple increment among the non-poor*** if:

$$x_{i'j} > y_{i'j} \quad \text{for} \quad (i, j) = (i', j') \quad \text{if} \quad i' \in Z$$

$$x_{i'j} = y_{i'j} \quad \text{for} \quad (i, j) \neq (i', j)$$

Poverty Focus: If X is obtained from Y by ***a simple increment among the non-poor***, then P(X;z)=P(Y;z).

Example of Poverty Focus

- Suppose person 3 is not considered multidimensionally poor, does it matter if he/she experiences an increase in any of the dimensions?

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Focus on deprived attributes

X is obtained from Y by a ***simple increment among the non-deprived*** if:

$$x_{i'j'} > y_{i'j'} \quad \text{for} \quad (i, j) = (i', j') \quad \text{if} \quad y_{i'j'} \geq z_{j'}$$

$$x_{ij} = y_{ij} \quad \text{if} \quad (i, j) \neq (i', j')$$

POVERTY FOCUS: If X is obtained from Y by ***a simple increment among the non-deprived***, then P(X;z)=P(Y;z).

Example of Deprivation Focus

- Suppose person 2 is considered multidimensionally poor, does it matter if he/she experiences an increase in health (dimension in which he/she is not deprived)

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad g^0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Focus

- One type of focus axiom does not necessarily imply the other one.

- Example: If a unidimensional approach was used to measure multidimensional poverty, assigning a price to each attribute, adding them up and comparing them with an income z, then an increment in a non-deprived attribute could compensate deprivation in another and reduce poverty, then poverty focus would be satisfied, but deprivation focus would not.

Continuity

P(X;z) is continuous if a small change in any attribute does not result in an abrupt change in the poverty index P(X;z)

- *CONTINUITY:* For any sequence X^k , if X^k converges to X, then $P(X^k;z)$ converges to $P(X;z)$

Monotonicity: Two types

- Becoming less deprived in a specific attribute (within dimension): Monotonicity

- Stop being deprived in one dimension (across dimensions): Dimensional Monotonicity

Monotonicity (within dimension)

X is obtained from Y by a ***deprived increment among the poor*** if for some *i*:

$x_{i'j'} > y_{i'j'}$ for $(i, j) = (i', j')$ where

$y_{i'j'} < z_{j'}$ and $i' \in Z$

while $x_{ij} = y_{ij}$ for all $(i, j) \neq (i', j')$

MONOTONICITY: If X is obtained from Y by a ***deprived increment*** among the poor, then $P(X,z)<P(Y,z)$.

Example of Monotonicity (within dimension)

- Suppose person 1 is considered multidimensionally poor, and experiences an improvement in his/her health.

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Dimensional Monotonicity (across dimensions)

X is obtained from Y by a ***dimensional increment among the poor*** if for some *i*:

$$x_{i'j'} \geq z_{j'} > y_{i'j'} \quad \text{for} \quad (i,j)=(i',j') \quad \text{where} \quad i' \in Z$$

$$\text{while} \quad x_{ij} = y_{ij} \quad \text{for} \quad \text{all} \quad (i,j) \neq (i',j')$$

DIMENSIONAL MONOTONICITY: If X is obtained from y by a ***dimensional increment among the poor***, then P(X,z)<P(Y,z).

Example of Dimensional Monotonicity (across dimension)

- Suppose person 2 is considered multidimensionally poor, and experiences an increment in his/her education such that now is no longer deprived in this dimension.

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 6 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

$$z = (5, 6, 4)$$

Extensions of the Pigou-Dalton Transfer Principle (I)

X is obtained from Y by **Uniform Pigou Dalton Transfer among the poor** (UPD) (an averaging of achievements between two poor people) if Y=TX, where T=λE+(1- λ)P (with 0< λ<1), with t_{ii}=1 for every non-poor person *i* in Y, E is an identity matrix and P is a permutation matrix.

$$T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = TY = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 & 3 \\ 3.5 & 4.5 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

The distribution of the attributes has been smoothed between individuals 1 and 2.

'd progressive transfers'

Extensions of the Pigou-Dalton Transfer Principle (I)

- *TRANSFER UNDER UPD :* If X is obtained from *Y* by a ***Uniform Pigou-Dalton Transfer*** among the poor (an averaging of achievements between two people), then $P(X;z) \leq P(Y;z)$.

Extensions of the Pigou-Dalton Transfer Principle (II-Kolm, 1977)

X is obtained from Y by multiplying it by a ***Uniform Majorization among the poor*** (an averaging of achievements among the poor) if X=BY, where B is a nxn bistochastic matrix but not a permutation matrix, and $b_{ii}=1$ for every non-poor person *i* in Y.

$$X = BY = \begin{bmatrix} 0.5 & 0.5 & \\ 0.5 & 0.5 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3.5 & 4.5 & 3 \\ 3.4 & 4.5 & 3 \\ 8 & 6 & 3 \end{bmatrix}$$

The distribution of the attributes has been smoothed among all individuals.

Extensions of the Pigou Dalton Transfer Principle (II)

- *Transfer under UM :* If X is obtained from *Y* a ***uniform majorization among the poor*** (an averaging of achievements among the poor), then $P(X;z) \leq P(Y;z)$.

UPD and UM

- Any T matrix is a B matrix, but the converse is not true. Then UDP implies UM but the converse is not true.

- Only when n=2 or k=1, UPD is equivalent to UM.

Axiom Specific to the Multidimensional Case

- Considers the ‘correlation’ between attributes when marginal distributions are unchanged (Atkinson & Bourguignon, 1982; Boland & Proschan, 1988).

- Intrinsic to the multivariate case.

Axiom Specific to the Multidimensional Case

- Example: A switch in dimension 3 between person 2 and 3

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix}$$

Ways to call this transfer:

- From Y to X: association increasing rearrangement, (Alkire & Foster, 2007).
- From X to Y: association decreasing rearrangement (Boland & Proschan, 1988); correlation-increasing transfer (Tsui, 1999), correlation increasing switch (Bourguignon & Chakravarty, 2003).

Axiom Specific to the Multidimensional Case

X is obtained from Y by an ***association decreasing rearrangement among the poor*** if for two persons *i* and *i':*

i) *i* and i' are poor in Y

ii) Either they have the same amount of attribute in X than in Y: $(x_{ij},x_{i'j})=(y_{ij},y_{i'j})$

iii) Or they have switched the amount of the attribute: $(x_{ij},x_{i'j})=(y_{i'j},y_{ij})$

iii) The amount of attributes of the other people $i''\neq i,i'$ remain unchanged: $x_{i''j}=y_{i''j}$

iv) Vectors of *i* and *i''* are comparable by vector dominance in y but not in x

Question…

- How do you think poverty should change under an association decreasing rearrangement?

Association Decreasing Rearrangement

$$Y = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 3 \\ 8 & 6 & 4 \end{bmatrix} \qquad X = \begin{bmatrix} 4 & 4 & 3 \\ 3 & 3 & 2 \\ 8 & 6 & 4 \end{bmatrix}$$

- If you think that a good health can ***substitute*** for a bad income or a bad education, then poverty should ***decrease***.
- If you think that a good health is ***necessary (complementary)*** to achieve a good income and a good education, then poverty should ***increase***.
- If you think that health is not necessary to achieve a good income and a good education, and can not either substitute for any of these, (ie: you think they are ***independent***), then poverty should ***not change***.

Axiom Specific to the Multidimensional Case

- *WEAK REARRANGEMENT :* If X is obtained from *Y* by an ***association decreasing rearrangement among the poor***, $P(X;z) \leq P(Y;z)$.

Axiom Specific to the Multidimensional Case

- What is the assumption behind the axiom?
  - Assumption: Attributes are <u>*independent*</u> (if =) or <u>*substitutes*</u> (if <) (compensating inequalities).
  - Could be <u>*complements*</u>, then axiom should go in the other way (>). (Bourguignon & Chakravarty, 2003).

ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

$U(x_1,x_2)$ such that:

- marginal utility for each attribute is non-decreasing: $$\frac{\partial U}{\partial x_1} \geq 0, \frac{\partial U}{\partial x_2} \geq 0$$

- If cross derivative is non-positive, attributes are substitutes: $$\frac{\partial^2 U}{\partial x_1 x_2} \leq 0$$

(Utility increases less with an increase in attribute 2 for persons with larger quantities of attribute 1).

- If cross derivative is non-negative, attributes are complements $$\frac{\partial^2 U}{\partial x_1 x_2} \geq 0$$

(Utility increases more with an increase in attribute 2 for persons with larger quantities of attribute 1).

ALEP Definition of Substitutes and Complements (Auspitz, Lieben, Edgeworth, Pareto)

- The cross derivative condition has to hold for the ‘least concave utility function’, so that the condition holds under monotonic transformations of the utility function. (Kannai, 1980).

Subgroup Consistency

- *Population Subgroup Consistency:*

  If $P(X_1;z)>P(Y_1;z)$ and $P(X_2;z)=P(Y_2;z)$, and $n(X_1)=n(Y_1)$, $n(X_2)=n(Y_2)$, then $P(X_1,Y_1)>P(X_2,Y_2;z)$.

Possible extension…

- *Dimension Subgroup Consistency:*

  If $P(X_1;z)>P(Y_1;z)$ and $P(X_2;z)=P(Y_2;z)$, and $d(X_1)=d(Y_1)$, $d(X_2)=d(Y_2)$, then $P(X_1,Y_1)>P(X_2,Y_2;z)$.

Decomposability

- *Population Subgroup Decomposability:*

  A poverty measure P is decomposable if
  $P(X,Y;z)=(n(X)/n)P(X)+(n(Y)/n)P(Y)$

- Extending this axiom to subgroups of dimensions in a strict way is not straightforward. It is called break-down by dimension, because it is dependent on the identification criterion of the multidimensionally poor. See Alkire & Foster (2007)

Example of Population Subgroups

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix}$$

	Income	Education	Health
Person 1	4	4	2
Person 2	3	5	4
Person 3	8	6	3

Example of Dimension Subgroups

Income Education Health

$$X = \begin{bmatrix} 4 & 4 & 2 \\ 3 & 5 & 4 \\ 8 & 6 & 3 \end{bmatrix} \begin{matrix} \textbf{Person 1} \\ \textbf{Person 2} \\ \textbf{Person 3} \end{matrix}$$