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Unidimensional Dominance

Poverty Orderings

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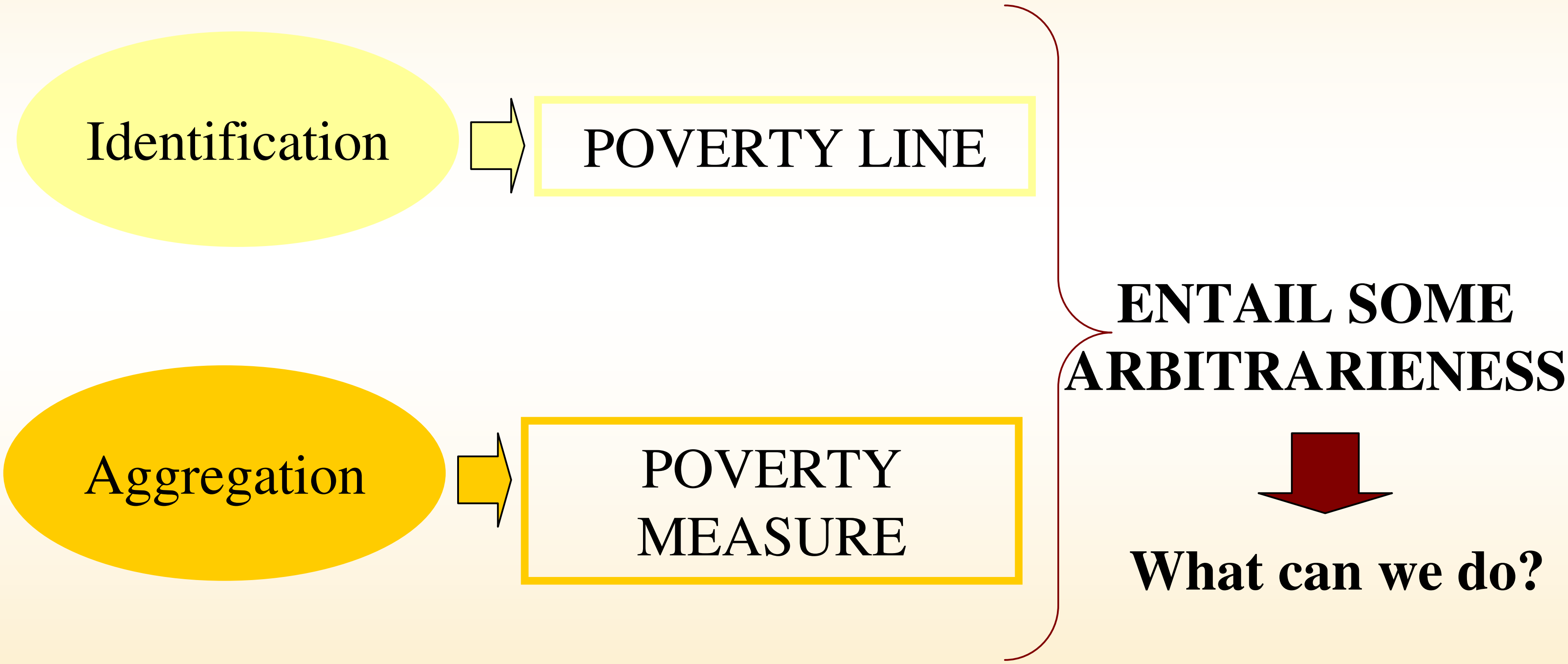
Main Sources of this Lecture

- Foster and Shorrocks (1988)
- Foster and Sen (1997), Annexe of “On Economic Inequality”.
- Atkinson (1987)
- There are others: please see the readings list.

When measuring poverty…



Example:

- Two states/provinces/regions in a country. Government needs to prioritize the poorest one.
- State A has 20% of poor people
- State B has 25% of poor people

How can we be sure that State B is indeed poorer than A?

Clearly, one wants to avoid the possibility of contradictory rankings at different poverty lines. And maybe also for different poverty measures.

Then, one needs to follow a **dominance approach**.

Dominance/(Quasi)Ordering Approach

- The quasi-ordering approach allows to determine when judgements hold for a *range* of poverty lines or a *class* of poverty measures.
- Then, the two steps become *robust* (just as the Lorenz criterion removes arbitrariness in the selection of an inequality measure).

Dominance/(Quasi)Ordering Approach

- Two main types of poverty orderings:

1. Variable-line poverty orderings (focus on the identification step)
2. Variable-measure poverty orderings (address aggregation).

It will turn out that these orderings are closely related with stochastic dominance relations.

Variable-line poverty orderings (Foster and Shorrocks, 1988)

- Main procedure:

1. Choose a measure
2. Identify the condition that two distributions must satisfy so as to be able to say that one is poorer than the other.

Definition of a strict partial ordering

x**P**y if and only if

$P(x;z) \leq P(y;z)$ for all z in R_{++} and

$P(x;z) < P(y;z)$ for some z in R_{++}

x**P**y means that x has *unambiguously* **less poverty than** y with respect to poverty index P.

FGT Poverty Orderings

- Foster and Shorrocks (1988) developed the conditions of poverty orderings for three members of the FGT family: P_0, P_1 and P_2.

Poverty Ordering based on P_0 (Headcount Ratio)

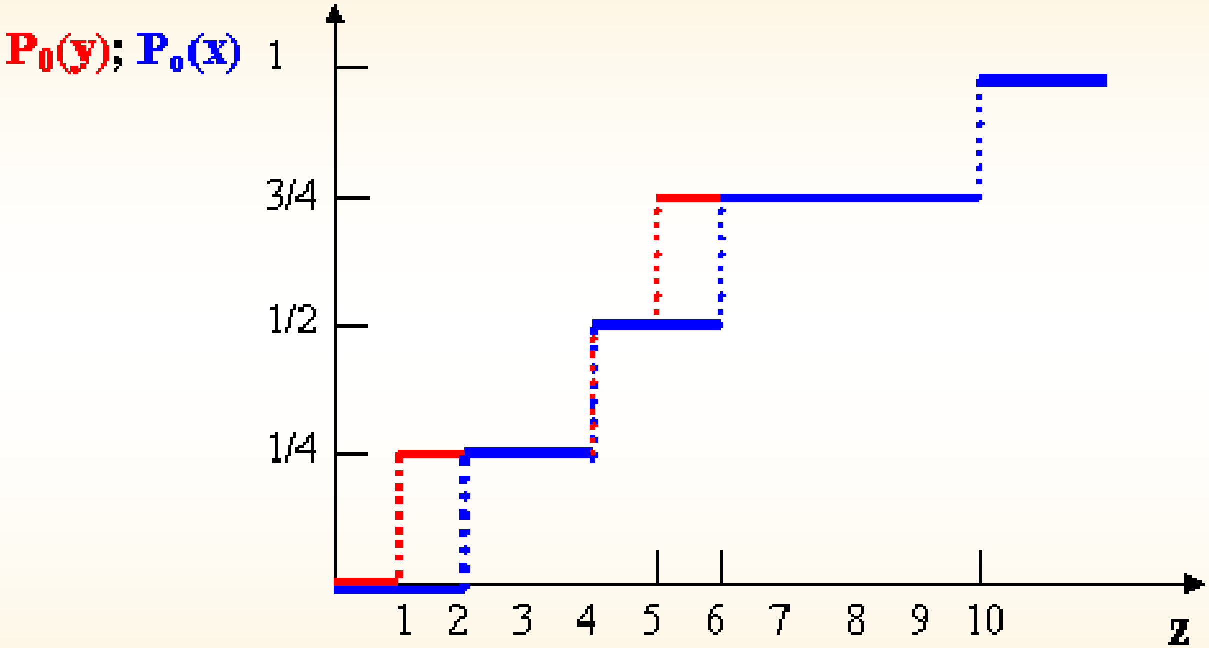
$$P_0(x;z) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{z - x_i^*}{z}\right)^0$$

- Intuitive way: to graph the poverty index as a function of the poverty line.
- Example: x=(2,4,6,10) y=(1,4,5,10). Poor if $x_i \leq z$
  $P_0(x;z) \leq P_0(y;z)$ for all z in R_{++}
  and < for some z in R_{++}
- What should happen graphically for one to have always less poverty than the other?

Z	$P_0(x)$	$P_0(y)$
1	0	¼
2	¼	¼
3	¼	¼
4	2/4	2/4
…		
10	1	1

Poverty Ordering based on P_0

- Example: x=(2,4,6,10) y=(1,4,5,10)



The graph of $P_0(x;z)$ must be nowhere above or to the left of $P_0(y;z)$.

Measured by Headcount Ratio, x has unambiguously less poverty than y.

Poverty Ordering based on P_0

- Note that the graph of the Headcount Ratio (P_0) is just the graph of the cdf!
- Also recall that this implies vector dominance (dominance element by element of the ordered vectors).

Poverty Ordering based on P_0

- For every integer $n \geq 1$ and any x, y in R^n_{++}, the following statements are equivalent:

1. $x\mathbf{P_0}y$
2. $\hat{x}_i \geq \hat{y}_i$ for all i=1,…,n and > for some I
3. $\hat{x}$ can be obtained from $\hat{y}$ by a finite nonempty sequence of simple increments.

(By replication invariance, the result also holds for distributions with different pop. size)

Poverty Ordering based on P_0

- The poverty ordering P_0 is identical to FSD:

x$\mathbf{P_0}$y if and only if x**FSD**y

(equivalently, x$\mathbf{W_0}$y)

(Foster and Shorrocks, 1988; Theorem 1)

Poverty Ordering based on P_0

- Then, there is a connection between the poverty ordering P_0 and a welfare ordering W_0, corresponding to welfare functions that are the sum (across people) of monotonically increasing individual utility functions.

- *"x has unambiguously less poverty than y by the headcount ratio"* implies that *x* is ranked better than *y* by all welfare functions in that class. Conversely, if all welfare functions in that class agree that *x* is better than *y*, there can be no poverty line at which the proportion of the population in poverty is higher in *x* than in *y*. (FS, p. 185)

Poverty Ordering based on P_1 (Per Capita Poverty Gap)

$$P_1(x;z) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{z - x_i^*}{z}\right)^1$$

- First note the link with P_0

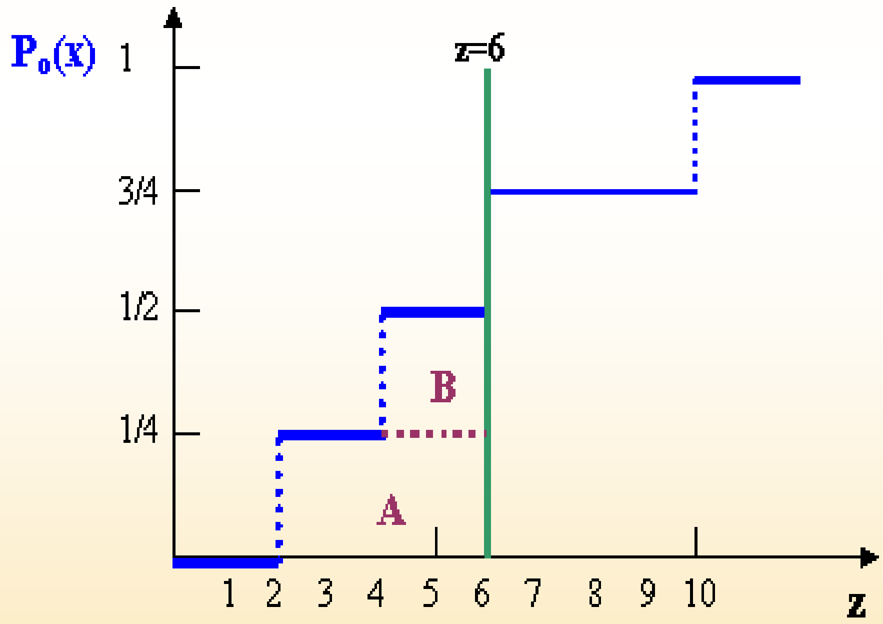
$$zP_1(x;z) = \int_0^z P_0(x;s)ds$$

- P_1 is the area beneath the graph of P_0 up to z.
- Example:

x=(2,4,6,10). Suppose z=6.

P_1=(1/4)[(6-2)/6+(6-4)/6]=

=(A+B)/6=0.25

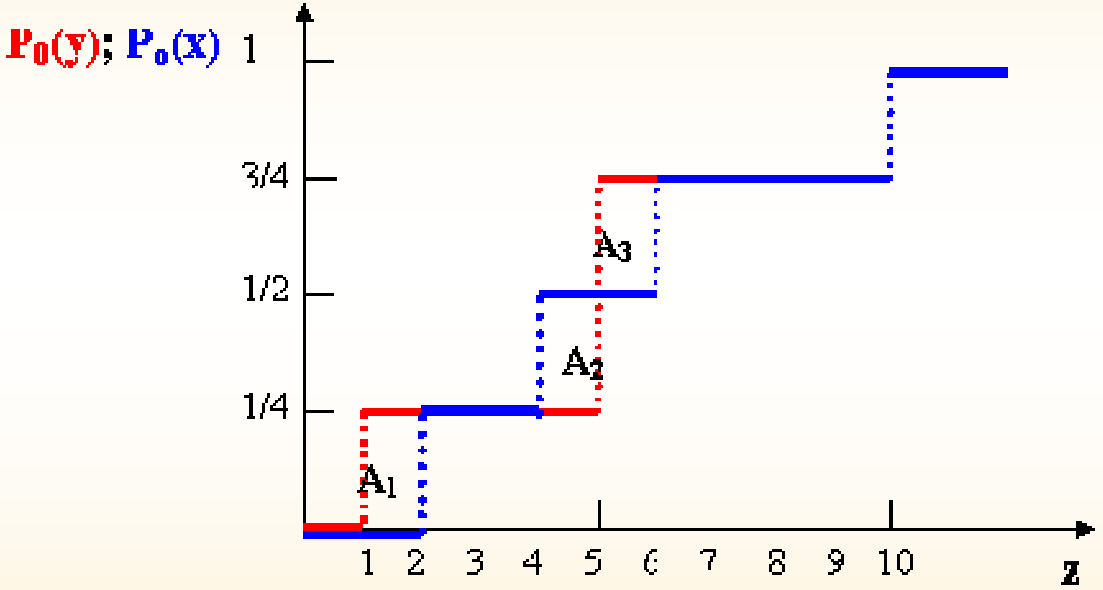


Poverty Ordering based on P_1

- Then, the P_1 ordering is such that…
  $P_1(x;z) \leq P_1(y;z)$ for all z in R_{++}
  and < for some z in R_{++}
- What should happen graphically for one to have always less poverty than the other according to P_1?

Poverty Ordering based on P_1

- Example: x=(2,4,6,10) y=(1,5,5,10).
- Clearly no P_0 ordering can be established now.



Integrating the areas under P0 should be no higher and for some z strictly lower for x than for y.

Measured by the Per Capita Poverty Gap, x has unambiguously less poverty than y.

Poverty Ordering based on P_1

- Note that this condition refers to the condition on integrating the cdfs!
- Also recall that this implies dominance of the sums of the lowest incomes in each distribution: the total income held by the poorest person, by the two poorest persons, etc, in x is no larger than in y.

Poverty Ordering based on P_1

- For every integer $n \geq 1$ and any x, y in R^n_{++}, the following statements are equivalent:

1. $x\mathbf{P_1}y$
2. $X_k := \sum_{i=1}^{k} \hat{x}_i \geq Y_k := \sum_{i=1}^{k} \hat{y}_i$ for all i=1,…,n and > for some k
3. $\hat{x}$ can be obtained from $\hat{y}$ by a finite nonempty sequence of simple increments and/or progressive transfers.

Poverty Ordering based on P_1

- The poverty ordering P_1 is identical to TSD:

$x\mathbf{P_1}y$ if and only if $x\mathbf{SSD}y$

(equivalent to $x\mathbf{W_1}y$ and $x\mathbf{GL}y$)

(Foster and Shorrocks, 1988; Theorem 2)

Poverty Ordering based on P_1

- The poverty ordering P1 is simply the generalized Lorenz dominance!
- Then, there is a connection between the poverty ordering P_1 and a welfare ordering $\mathbf{W_1}$, corresponding to welfare functions that are the sum (across people) of individual utility functions with positive and decreasing marginal utility (ie. such that progressive transfers improve welfare).

Poverty Ordering based on P_1

- Then, unanimous welfare verdicts for the class of symmetric, monotonic and equality preferring welfare functions correspond to unambiguous poverty veredicts according to the per capita poverty gap. Moreover, the GL equivalence offers a simple implementation rule for P_1: just check whether the GL curves intersect. If they don't, then the higher GL curve has unambiguously less poverty, as measured by P_1. (Foster &Shorrocks, 1988, p. 188).

Poverty Ordering based on P_2

$$P_2(x;z) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{z - x_i^*}{z}\right)^2$$

- Link with P_1 and P_0

$$\frac{1}{2}z^2 P_2(x;z) = \int_0^z sP_1(x;s)ds = \int_0^z \int_0^t P_0(x;t)dtds$$

- So P_2 is associated with double integrating a cdf… which order of dominance does this remind of?

Poverty Ordering based on P_2

- Definition of a Favourable Composite Transfer ‘FACT’: a progressive transfer at the lower end of the distribution and a regressive transfer at the upper end, such that variance does not change (effects cancel out) (Shorrocks and Foster 1987)

Poverty Ordering based on P_2

- For every integer $n \geq 1$ and any x, y in R^n_{++}, the following statements are equivalent:

1. $x\mathbf{P_2}y$

2. $\hat{x}$ can be obtained from $\hat{y}$ by a finite nonempty sequence of simple increments and/or progressive transfers and/or FACTs.

Poverty Ordering based on P_2

- The poverty ordering P_2 is identical to SSD:

$x\mathbf{P_2}y$ if and only if $x\mathbf{TSD}y$

(equivalently $x\mathbf{W_2}y$)

(Foster and Shorrocks, 1988; Theorem 3)

Poverty Ordering based on P_2

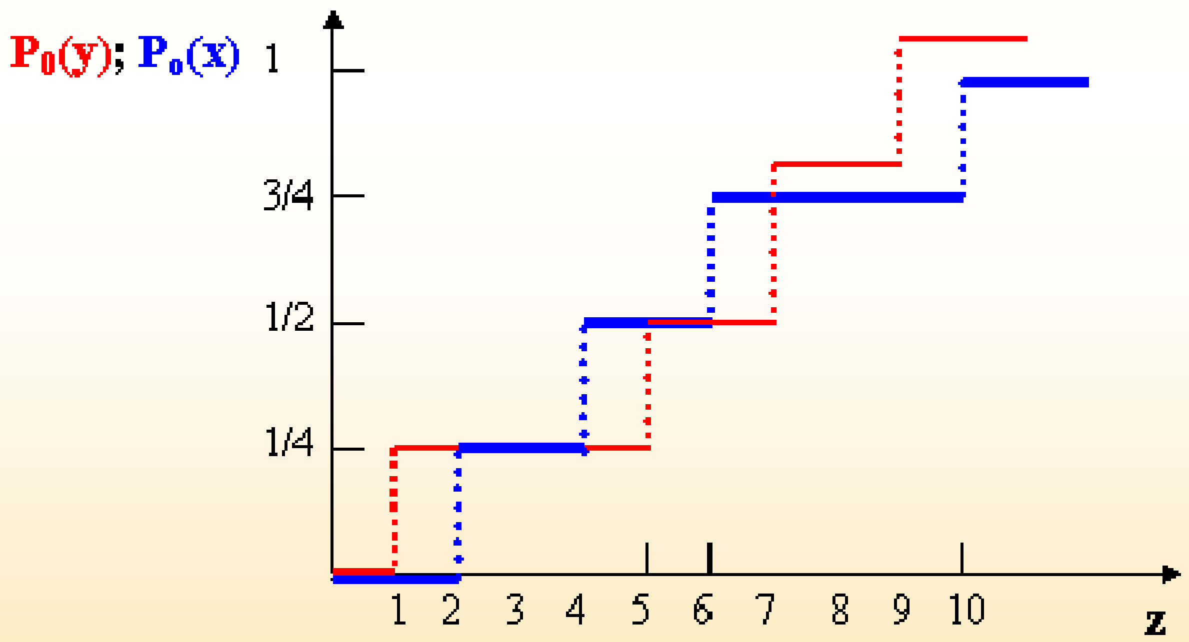
- Then, there is a connection between the poverty ordering P_2 and a welfare ordering $\mathbf{W_2}$, corresponding to welfare functions that are the sum (across people) of individual utility functions with positive, decreasing and marginal utility (which implies that, given a progressive transfer, the improvement in welfare will be lower the higher the income of the person that received the transfer.

Poverty Ordering based on P_2

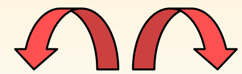
- For this class of welfare measures, the positive welfare impact of any progressive transfer will always outweigh the negative welfare impact of a comparable regressive transfer taking place at a higher income level.

Poverty Ordering based on P_2

- Example: x=(2,4,6,10) y=(1,5,7,9)
- Clearly no FSD, no SSD.



Poverty Ordering based on P_2



- x=(2,4,6,10) y=(1,5,7,9)
- Var(x)=Var(y)=8.75
- Then x can be obtained from y by a FACT.
- TSD by vector comparison:

(First: accumulate the original ordered vector elements x’=(2,6,12,22) y’=(1,6,13,22) and then compare the accumulation of those:

2>1; 2+6>1+6; 2+6+12=1+6+13; 2+6+12+22=1+6+13+22

Limited Range Poverty Orderings

- While deciding the precise value of the poverty line may be difficult, agreement is likely to occur on an interval Z.
- So now the poverty ordering would be defined as x**P(Z)**y when

$$P(y;z) \geq P(x;z) \text{ for all } z \text{ in } Z$$

$$\text{and } > \text{ for some } z \text{ in } Z.$$

Limited Range Poverty Orderings

- By restricting the values of z, the obtained ranking P(Z) will be "more complete" than the P ranking (but less general).
- Indeed, looking at extremely high poverty lines, does not make sense. So now we are setting an upper bound z^*, so that the relevant range is $Z^*=(0,z^*)$, and $\mathbf{P^*_\alpha}$ being the poverty ordering.

Limited Range Poverty Orderings

- Then, one can work with the censored distribution, 'ignoring' incomes above z*, ie: replacing them by z*:

$$x_i(z^*)=\min(x_i,z^*) \text{ for } i=1,\ldots,n(x)$$

- Define ordering W^*_α:

$$x\ \mathbf{W}^*_{\boldsymbol{\alpha}} y \quad \text{iff} \quad x(z^*)\mathbf{W}^*_{\boldsymbol{\alpha}} y(z^*)$$

Limited Range Poverty Orderings

- For any x, y in R^n_{++} (for some finite n):

1. $x\ \mathbf{W}^*_{\mathbf{0}} y$ iff $x \mathbf{P}^*_{\mathbf{0}} y$
2. $x\ \mathbf{W}^*_{\mathbf{1}} y$ iff $x \mathbf{P}^*_{\mathbf{1}} y$
3. $x\ \mathbf{W}^*_{\mathbf{2}} y$ iff $x \mathbf{P}^*_{\mathbf{2}} y$ and $P_1(x,z^*) \leq P_1(y,z^*)$

Limited Range Poverty Orderings

- Therefore, the ‘limited range’ poverty orderings are simply the unlimited poverty orderings applied to the distributions censored at z*.
- In particular, when the gap measure P_1 is used, then x***GL**y* indicates that x has less poverty than y at some line below z* and no higher poverty at all lines.
- In other words, the limited range poverty ordering for the gap measure is simply the censored generalized Lorenz ranking GL*, where x**GL***y is defined by x***GL**y* .

Link btw variable line and variable measure poverty orderings

- Atkinson (1987) established a fundamental connection between the *variable-poverty lines* orderings and *variable-measure* orderings.

Link btw variable line and variable measure poverty orderings

- If xGL*y (with x and y same n), then x* can be obtained from y* by a combination of permutations, progressive transfers and increments among the poor.

  Then, suppose that P is a continuous poverty measure satisfying symmetry, monotonicity, focus, transfer and replication invariance: $P(x^*;z^*)<P(y^*;z)$.
- Replication invariance extends the conclusion to distributions with arbitrary n.

Link btw variable line and variable measure poverty orderings

- Thus xGL*y (second order stochastic dominance up to z*) implies that y has more poverty than x:

1. Across *all* poverty lines below z*
2. according to *all* continuous poverty measures satisfying symmetry, monotonicity, focus, transfer and replication invariance (and the converse also holds).

Link btw variable line and variable measure poverty orderings

- Additionally, if first order dominance up to z* holds, then there is agreement for *all* continuous poverty measures satisfying symmetry, monotonicity, focus, and replication invariance. Note that transfer is not needed in this case.