A note on the standard errors of the members of the Alkire Foster family and its components

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Comments and corrections welcome.

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1 Basic notation

Consider a matrix X, whose N rows have information on the attainments of N inviduals. Each column, therefore, hosts the distribution of each attainment across the population. The number of columns/variables is D. A typical attainment element of the matrix is: $x_{nd}\,(\in \mathbb{R})$, that is, the attainment of individual n in dimension/variable d.

In the identification stage, the variable-specific poverty lines are denoted by z_d;[1] and for the second identification stage, i.e. to determine who is multidimensionally poor, variables are weighted by weights w_d such that: $w_d \in \mathbb{R}_+ \wedge \sum_d^D w_d = D$. The matrix of deprivations is formed by replacing x_{nd} with a deprivation gap, g_{nd}, such that:

$$\begin{aligned} g_{nd}(k) &= \frac{z_d - x_{nd}}{z_d} \text{ if } z_d > x_{nd} \wedge c_n \geq k \\ g_{nd}(k) &= 0 \text{ otherwise} \end{aligned} \tag{1}$$

where $k \leq D$ is the multidimensional-deprivation cut-off and c_n is the weighted number of deprivations suffered by individual n. If $c_n \geq k$ individual n is said, and identified, to be multidimensionally poor. $c_n \equiv \sum_{d=1}^D w_d I(z_d > x_{nd})$ [2]

Now the multidimensional headcount can be defined:

$$H(X;k,Z) \equiv \frac{1}{N}\sum_{n=1}^{N}\left[\sum_{d=1}^{D} w_d g_{nd}(k)\right]^0 = \frac{1}{N}\sum_{n=1}^{N} I(c_n \geq k) \tag{2}$$

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[1] From a vector of poverty lines, $Z : (z_1, \ldots, z_d, \ldots, z_D)$.

[2] $I()$ is an indicator that takes the value of 1 if the expression in parenthesis is true. Otherwise it takes the value of 0.

Also the average number of deprivations of the multidimensionally poor in the same period is defined:

$$A(X;k,Z) \equiv \frac{\sum_{n=1}^{N}\sum_{d=1}^{D} w_d \left[g_{nd}(k)\right]^0}{D\sum_{n=1}^{N}\left[\sum_{d=1}^{D} w_d g_{nd}(k)\right]^0} = \frac{\sum_{n=1}^{N} I(c_n \geq k)\, c_n}{DNH(X;k,Z)} \tag{3}$$

Finally the adjusted headcount ratio is:

$$M^0(X;k,Z) \equiv H(X;k,Z)\,A(X;k,Z) = \frac{\sum_{n=1}^{N} I(c_n \geq k)\, c_n}{DN} \tag{4}$$

More generally, the Alkire-Foster family can be defined as follows:

$$\mathcal{AF} : \{M^\alpha(X;k,Z) \equiv \frac{\sum_{n=1}^{N}\sum_{d=1}^{D} w_d g_{nd}^\alpha}{DN}\ \forall \alpha \in \mathbb{N}\}. \tag{5}$$

When $\alpha = 0 : M^0 = \frac{\sum_{n=1}^{N} c_n}{DN}$, as in (4). A more useful definition of $M^\alpha(X;k,Z)$, for the purposes of deriving standard errors is the following:

$$M^\alpha(X;k,Z) \equiv \frac{1}{N}\sum_{n=1}^{N} I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right) \forall \alpha \in \mathbb{N}, \tag{6}$$

where $\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha = \left[\frac{z_d - x_{nd}}{z_d}\right]^\alpha$ if $z_d > x_{nd}$. Otherwise: $\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha = 0$.

2 The case of a simple random sample

Since $M^\alpha(X;k,Z)$ is the average of $I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right)$ across the population. Then under the assumption that each value, $I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right)$, is identically and independently distributed across the population, then the result follows:

$$\sqrt{N}\left(M^\alpha(X;k,Z) - \mu^\alpha(X;k,Z)\right) \xrightarrow{d} N\left(0, \sigma_\alpha^2\right), \tag{7}$$

where $\mu^\alpha(X;k,Z) \equiv E\left[I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right)\right]$

and $\sigma_\alpha^2 \equiv E\left[I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right) - \mu^\alpha(X;k,Z)\right]^2$. The empirical counterparts are the following:

$$\widehat{\sigma_\alpha^2} = \frac{1}{N}\sum_{n=1}^{N}\left[I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^\alpha\right) - M^\alpha(X;k,Z)\right]^2 \tag{8}$$

Therefore the standard error of $M^\alpha(X;k,Z)$ is:

$$SE\left(M^{\alpha}\left(X;k,Z\right)\right)=\sqrt{\frac{\widehat{\sigma_{\alpha}^{2}}}{N}} \tag{9}$$

In the case of $H\left(X;k,Z\right)$ a similar reasoning applies,

i.e. $\sqrt{N}\left(H\left(X;k,Z\right)-\eta\left(X;k,Z\right)\right)\overset{d}{\longrightarrow}N\left(0,\sigma_{H}^{2}\right)$,where $\eta\left(X;k,Z\right)\equiv E\left[I\left(c_{n}\geq k\right)\right]$. So its standard error is:

$$SE\left(H\left(X;k,Z\right)\right)=\sqrt{\frac{\widehat{\sigma_{H}^{2}}}{N}}, \tag{10}$$

where:

$$\widehat{\sigma_{H}^{2}}=\frac{1}{N}\sum_{n=1}^{N}\left[I\left(c_{n}\geq k\right)-H\left(X;k,Z\right)\right]^{2}=H\left(X;k,Z\right)\left[1-H\left(X;k,Z\right)\right] \tag{11}$$

The case of $A\left(X;k,Z\right)$ is less straightforward because: $A\left(X;k,Z\right)=\frac{M^{0}(X;k,Z)}{H(X;k,Z)}$.Hence A does not have an exact standard error, but an asymptotic one. Deriving it require a first-order Taylor expansion of $A\left(X;k,Z\right)$ around $\frac{\mu^{0}(X;k,Z)}{\eta(X;k,Z)}$:

$$\begin{aligned}A\left(X;k,Z\right)-\frac{\mu^{0}\left(X;k,Z\right)}{\eta\left(X;k,Z\right)} &\cong \frac{1}{H\left(X;k,Z\right)}\left(M^{0}\left(X;k,Z\right)-\mu^{0}\left(X;k,Z\right)\right)\\ &\quad-\frac{M^{0}\left(X;k,Z\right)}{\left[H\left(X;k,Z\right)\right]^{2}}\left(H\left(X;k,Z\right)-\eta\left(X;k,Z\right)\right)\end{aligned} \tag{12}$$

From (7) and its equivalent for $H\left(X;k,Z\right)$, and from (12), it can be deduced that:

$$\sqrt{N}\left(A\left(X;k,Z\right)-\frac{\mu^{0}\left(X;k,Z\right)}{\eta\left(X;k,Z\right)}\right)\overset{d}{\longrightarrow}N\left(0,\sigma_{A}^{2}\right), \tag{13}$$

where σ_{A}^{2} is the asymptotic variance of the left-hand sife of (13):

$$\sigma_{A}^{2}=\frac{1}{\left[H\left(X;k,Z\right)\right]^{2}}\sigma_{\alpha}^{2}+\left(\frac{M^{0}\left(X;k,Z\right)}{\left[H\left(X;k,Z\right)\right]^{2}}\right)^{2}\sigma_{H}^{2}-2\frac{M^{0}\left(X;k,Z\right)}{\left[H\left(X;k,Z\right)\right]^{3}}\sigma_{\alpha,H}, \tag{14}$$

where $\sigma_{\alpha,H}$ is the covariance of $I\left(c_{n}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}}{z_{d}}\right]_{+}^{\alpha}\right)$ and $I\left(c_{n}\geq k\right)$. To estimate the empirical counterpart of σ_{A}^{2}, $\widehat{\sigma_{A}^{2}}$, the empirical counterparts for σ_{α}^{2} and σ_{H}^{2}, and the covariance, $\sigma_{\alpha,H}$, are needed. The first two are in (8) and (11), respectively. The covariance's it is:

$$\begin{aligned}\widehat{\sigma_{\alpha,H}} &= \frac{1}{N}\sum_{n=1}^{N}\left[I\left(c_{n}\geq k\right)-H\left(X;k,Z\right)\right]\left[I\left(c_{n}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}}{z_{d}}\right]_{+}^{\alpha}\right)-M^{\alpha}\left(X;k,Z\right)\right]\\ &= \frac{1}{N}\sum_{n=1}^{N}\left[I\left(c_{n}\geq k\right)^{2}\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}}{z_{d}}\right]_{+}^{\alpha}\right)\right]-H\left(X;k,Z\right)M^{\alpha}\left(X;k,Z\right)\end{aligned} \tag{15}$$

And finally the asymptotic standard error of A is:

$$SE\left(A\left(X;k,Z\right)\right)=\sqrt{\frac{\widehat{\sigma_{A}^{2}}}{N}}. \tag{16}$$

2.1 Calculating percentage changes using cross-sectional data

Defining the percentage change of any of the above statistics requires indexing the matrix of attainments by a time period, e.g. t or $t-a$. In the case of cross-sectiona data, the population sizes in each different period also need to be indexed accordingly. Then, for instance, the percentage change of any member of the Alkire-Foster family is:

$$\Delta\%M_{a}^{\alpha}\left(X^{t},X^{t-a};k,Z\right)\equiv\frac{M^{\alpha}\left(X^{t};k,Z\right)-M^{\alpha}\left(X^{t-a};k,Z\right)}{M^{\alpha}\left(X^{t-a};k,Z\right)}=\frac{M^{\alpha}\left(X^{t};k,Z\right)}{M^{\alpha}\left(X^{t-a};k,Z\right)}-1 \tag{17}$$

Since $\Delta\%M_{a}^{\alpha}$ is also made of the ratio of two averages, its standard error is also asymptotic and is derived like A's. although in the case of cross-sectional data, $M^{\alpha}\left(X^{t};k,Z\right)$ and $M^{\alpha}\left(X^{t-a};k,Z\right)$ are independent.

In the case of $\Delta\%M_{a}^{\alpha}\left(X^{t},X^{t-a};k,Z\right)$, the required first-order Taylor expansion is:

$$\begin{aligned}\frac{M^{\alpha}\left(X^{t};k,Z\right)}{M^{\alpha}\left(X^{t-a};k,Z\right)}-\frac{\mu^{\alpha}\left(X^{t};k,Z\right)}{\mu^{\alpha}\left(X^{t-a};k,Z\right)} \cong\ & \frac{1}{M^{\alpha}\left(X^{t-a};k,Z\right)}\left(M^{\alpha}\left(X^{t};k,Z\right)-\mu^{\alpha}\left(X^{t};k,Z\right)\right) \\ & -\frac{M^{\alpha}\left(X^{t};k,Z\right)}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\left(M^{\alpha}\left(X^{t-a};k,Z\right)-\mu^{\alpha}\left(X^{t-a};k,Z\right)\right)\end{aligned} \tag{18}$$

Resorting to the same reasoning used to derive the asymptotic standard error of A, i.e. considering the equivalent of expressions (12) through (16) applied to (18) yield the following results:

$$\sqrt{\frac{N^{t}N^{t-a}}{N^{t}+N^{t-a}}}\left(\frac{M^{\alpha}\left(X^{t};k,Z\right)}{M^{\alpha}\left(X^{t-a};k,Z\right)}-\frac{\mu^{\alpha}\left(X^{t};k,Z\right)}{\mu^{\alpha}\left(X^{t-a};k,Z\right)}\right)\xrightarrow{d}N\left(0,\sigma_{\Delta\%M}^{2}\right), \tag{19}$$

where $\sigma_{\Delta\%M}^{2}$ is the asymptotic variance of the left-hand sife of (19):

$$\sigma_{\Delta\%M^{\alpha}}^{2}=\frac{1}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\sigma_{\alpha(t)}^{2}\lambda+\left(\frac{M^{\alpha}\left(X^{t};k,Z\right)}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\right)^{2}\sigma_{\alpha(t-a)}^{2}\left[1-\lambda\right] \tag{20}$$

where $\sigma_{\alpha(t)}^{2}$ is the variance of $I\left(c_{n}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}}{z_{d}}\right]_{+}^{\alpha}\right)$ in period t, $\sigma_{\alpha(t-a)}^{2}$ is the respective variance for period $t-a$; and $\lambda=\frac{N^{t-a}}{N^{t}+N^{t-a}}$. Notice that the covariance element is absent from the right-side of (20). That is the case because $M^{\alpha}\left(X^{t};k,Z\right)$ and $M^{\alpha}\left(X^{t-a};k,Z\right)$ are independent.

And finally the asymptotic standard error of $\Delta\%M_{a}^{\alpha}$ is:

$$SE\left(\Delta\%M_{a}^{\alpha}\left(X^{t},X^{t-a};k,Z\right)\right)=\sqrt{\widehat{\sigma_{\Delta\%M^{\alpha}}^{2}}\left(\frac{1}{N^{t}}+\frac{1}{N^{t-a}}\right)}. \tag{21}$$

2.2 Calculating percentage changes using panel data

With panel data the matrices of attainments for the two periods stem from the same observations. Therefore, unlike the case of cross-sectional data, $M^{\alpha}\left(X^{t};k,Z\right)$ and $M^{\alpha}\left(X^{t-a};k,Z\right)$ are no longer independent. The same first-order Taylor expansion as in (18) is useful in this context. But now the results are the following:

$$\sqrt{N}\left(\frac{M^{\alpha}\left(X^{t};k,Z\right)}{M^{\alpha}\left(X^{t-a};k,Z\right)}-\frac{\mu^{\alpha}\left(X^{t};k,Z\right)}{\mu^{\alpha}\left(X^{t-a};k,Z\right)}\right)\xrightarrow{d}N\left(0,\sigma^{2}_{\Delta\%M}\right), \tag{22}$$

where now the asymptotic variance, $\sigma^{2}_{\Delta\%M}$, is similar to that of expression (14):

$$\sigma^{2}_{\Delta\%M}=\frac{1}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\sigma^{2}_{\alpha(t)}+\left(\frac{M^{\alpha}\left(X^{t};k,Z\right)}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\right)^{2}\sigma^{2}_{\alpha(t-a)}-2\frac{M^{\alpha}\left(X^{t};k,Z\right)}{\left[M^{\alpha}\left(X^{t-a};k,Z\right)\right]^{2}}\sigma_{\alpha(t),\alpha(t-a)}. \tag{23}$$

The covariance between the two measures in different periods, $\sigma_{\alpha(t),\alpha(t-a)}$, is:

$$\begin{aligned}\widehat{\sigma_{\alpha(t),\alpha(t-a)}} &= \frac{1}{N}\sum_{n=1}^{N}\left[I\left(c_{n}^{t}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}^{t}}{z_{d}}\right]_{+}^{\alpha}\right)-M^{\alpha}\left(X^{t};k,Z\right)\right] \\ &\quad \left[I\left(c_{n}^{t-a}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}^{t-a}}{z_{d}}\right]_{+}^{\alpha}\right)-M^{\alpha}\left(X^{t-a};k,Z\right)\right] \\ &= \frac{1}{N}\sum_{n=1}^{N}\left[I\left(c_{n}^{t}\geq k\right)I\left(c_{n}^{t-a}\geq k\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}^{t}}{z_{d}}\right]_{+}^{\alpha}\right)\left(\sum_{d=1}^{D}\frac{w_{d}}{D}\left[\frac{z_{d}-x_{nd}^{t-a}}{z_{d}}\right]_{+}^{\alpha}\right)\right] \\ &\quad -M^{\alpha}\left(X^{t};k,Z\right)M^{\alpha}\left(X^{t-a};k,Z\right)\end{aligned} \tag{24}$$

Finally the asymptotic standard error of $\Delta\%M_{a}^{\alpha}$ for the panel-data case is:

$$SE\left(\Delta\%M_{a}^{\alpha}\left(X^{t},X^{t-a};k,Z\right)\right)=\sqrt{\frac{\widehat{\sigma^{2}_{\Delta\%M^{\alpha}}}}{N}}. \tag{25}$$

3 The case of a two-stage, stratified household survey (following Deaton (1997))

This section derives the standard errors for the main statistics of the Alkire-Foster family, combining the above results with the formulas provided by Deaton (1997) in order to estimate means and variances of means from two-stage, stratified samples.

3.1 Basic notation

The data are assumed to come from a stratified sample. Following the notation by Deaton (1997), there are S strata each subindexed by s. Within each stratum households are drawn in two stages. In the first stage n_s clusters are drawn in each stratum separately. Then in the second stage m_c households are drown in every cluster, each indexed by i. The respective Alkire-Foster statistics are indexed accordingly. For instance, M_{sc}^0 is the adjusted headcount ratio of cluster c from stratum s. The data also come accompanied by weights, w, each subindexed as corresponds (not to be confused with dimension weights, w_d). These weights are inverse to the probability of being sampled into the dataset.

The multidimensional headcount is now the following:

$$H\left(X;k,Z\right)\equiv\frac{1}{\widehat{N}}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left(\sum_{i=1}^{m_c}w_{ics}\left[\sum_{d=1}^{D}w_d g_{id}\left(k\right)\right]^0\right)=\frac{1}{\widehat{N}}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left(\sum_{i=1}^{m_c}w_{ics}I\left(c_i\geq k\right)\right) \tag{26}$$

where:

$$\widehat{N}=\sum_{s=1}^{S}\sum_{c=1}^{n_s}\sum_{i=1}^{m_c}w_{ics}$$

The average number of deprivations of the multidimensionally poor is:

$$A\left(X;k,Z\right)\equiv\frac{1}{D\widehat{N}H}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left(\sum_{i=1}^{m_c}w_{ics}I\left(c_i\geq k\right)c_i\right) \tag{27}$$

The adjusted headcount ratio is:

$$M^0\left(X;k,Z\right)\equiv H\left(X;k,Z\right)A\left(X;k,Z\right)=\frac{1}{D\widehat{N}}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left(\sum_{i=1}^{m_c}w_{ics}I\left(c_i\geq k\right)c_i\right) \tag{28}$$

And the Alkire-Foster family is now defined as follows:

$$\mathcal{AF}:\{M^\alpha\left(X;k,Z\right)\equiv\frac{\sum_{s=1}^{S}\sum_{c=1}^{n_s}\sum_{i=1}^{m_c}w_{ics}\left[\sum_{d=1}^{D}w_d g_{id}^\alpha\right]}{D\widehat{N}}\ \forall\alpha\in\mathbb{N}\}. \tag{29}$$

An alternative definition of $M^\alpha\left(X;k,Z\right)$, for the purposes of deriving standard errors is the following:

$$M^\alpha\left(X;k,Z\right)\equiv\frac{1}{\widehat{N}}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left[\sum_{i=1}^{m_c}w_{ics}I\left(c_i\geq k\right)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{id}}{z_d}\right]_+^\alpha\right)\right]\forall\alpha\in\mathbb{N}, \tag{30}$$

3.2 Standard errors

The variance of $M^\alpha\left(X;k,Z\right)$ can be computed using Deaton's formula (1.63) (Deaton, 1997, p. 56). The formula is:

$$\widehat{var_\alpha} = \frac{1}{\widehat{N}^2}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left[\left(\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}}{z_d}\right]_+^\alpha - M_s^\alpha\right) - M^\alpha\left(w_{cs} - \overline{w_s}\right)\right]^2, \tag{31}$$

where $M_s^\alpha = \frac{\sum_{c=1}^{n_s}\sum_{i=1}^{m_c} w_{ics} I(c_i \geq k)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}}{z_d}\right]_+^\alpha}{\sum_{c=1}^{n_s}\sum_{i=1}^{m_c} w_{ics}}$, $w_{cs} = \sum_{i=1}^{m_c} w_{ics}$, and $\overline{w_s} = \frac{1}{n_s}\sum_{c=1}^{n_s} w_{cs}$. Then the standard error of $M^\alpha(X;k,Z)$ is: $SE\left(M^\alpha(X;k,Z)\right) = \sqrt{\widehat{var_\alpha}}$.

In the case of $H(X;k,Z)$, its standard error is:$SE\left(H(X;k,Z)\right) = \sqrt{\widehat{var_H}}$ where:

$$var_H = \frac{1}{\widehat{N}^2}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left[\left(\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right) - H_s^\alpha\right) - H\left(w_{cs} - \overline{w_s}\right)\right]^2, \tag{32}$$

and $H_s^\alpha = \frac{\sum_{c=1}^{n_s}\sum_{i=1}^{m_c} w_{ics} I(c_i \geq k)}{\sum_{c=1}^{n_s}\sum_{i=1}^{m_c} w_{ics}}$. In the case of $A(X;k,Z)$, the asymptotic standard error is based on the approximation (12). The asymptotic variance is:

$$\widehat{var_A} = \frac{1}{\left[H(X;k,Z)\right]^2}\widehat{var_\alpha} + \left(\frac{M^0(X;k,Z)}{\left[H(X;k,Z)\right]^2}\right)^2\widehat{var_H} - 2\frac{M^0(X;k,Z)}{\left[H(X;k,Z)\right]^3}\widehat{covar_{\alpha,H}}, \tag{33}$$

where $\widehat{covar_{\alpha,H}}$ is the empirical counterpart of the covariance of $\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}}{z_d}\right]_+^\alpha$ and $\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right)$:

$$\begin{aligned}\widehat{\sigma_{\alpha,H}} \;=\; & \frac{1}{\widehat{N}^2}\sum_{s=1}^{S}\sum_{c=1}^{n_s}\left[\left(\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}}{z_d}\right]_+^\alpha - M_s^\alpha\right) - M^\alpha\left(w_{cs} - \overline{w_s}\right)\right] \\ & \left[\left(\sum_{i=1}^{m_c} w_{ics} I\left(c_i \geq k\right) - H_s^\alpha\right) - H^\alpha\left(w_{cs} - \overline{w_s}\right)\right]\end{aligned} \tag{34}$$

And finally the asymptotic standard error of A is:

$$SE\left(A(X;k,Z)\right) = \sqrt{\widehat{\sigma_A^2}}. \tag{35}$$

3.3 Calculating percentage changes using cross-sectional data

The asymptotic variance of $\Delta\% M_a^\alpha\left(X^t, X^{t-a}; k, Z\right)$ is also based on (18):

$$\widehat{var_{\Delta\% M^\alpha}} = \frac{1}{\left[M^\alpha\left(X^{t-a};k,Z\right)\right]^2}\widehat{var_{\alpha(t)}} + \left(\frac{M^\alpha\left(X^t;k,Z\right)}{\left[M^\alpha\left(X^{t-a};k,Z\right)\right]^2}\right)^2\widehat{var_{\alpha(t-a)}} \tag{36}$$

where $\widehat{var_{\alpha(t)}}$ is the variance $\widehat{var_\alpha}$ in period t, $\sigma^2_{\alpha(t-a)}$, $\widehat{var_{\alpha(t-a)}}$ is the respective variance for period $t-a$. The asymptotic standard error of $\Delta\% M_a^\alpha$ is:

$$SE\left(\Delta\% M_a^\alpha\left(X^t, X^{t-a}; k, Z\right)\right) = \sqrt{\widehat{var_{\Delta\% M^\alpha}}}. \tag{37}$$

3.4 Calculating percentage changes using panel data

The asymptotic variance, $\widehat{var_{\Delta\%M^\alpha}}$, is now:

$$\widehat{var_{\Delta\%M^\alpha}} = \frac{1}{\left[M^\alpha\left(X^{t-a};k,Z\right)\right]^2}\widehat{var_{\alpha(t)}} + \left(\frac{M^\alpha\left(X^t;k,Z\right)}{\left[M^\alpha\left(X^{t-a};k,Z\right)\right]^2}\right)^2 \widehat{var_{\alpha(t-a)}} - 2\frac{M^\alpha\left(X^t;k,Z\right)}{\left[M^\alpha\left(X^{t-a};k,Z\right)\right]^2}\widehat{\sigma_{\alpha(t),\alpha(t-a)}}. \tag{38}$$

The covariance between the two measures in different periods, $\widehat{\sigma_{\alpha(t),\alpha(t-a)}}$, is:

$$\begin{aligned}\widehat{\sigma_{\alpha(t),\alpha(t-a)}} = \frac{1}{\widehat{N}^2}\sum_{s=1}^{S}\sum_{c=1}^{n_s} & \left[\begin{array}{c}\left(\sum_{i=1}^{m_c} w_{ics}^t I\left(c_i^t \geq k\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}^t}{z_d}\right]_+^\alpha - M_s^\alpha\left(X^t;k,Z\right)\right) \\ -M^\alpha\left(X^t;k,Z\right)\left(w_{cs}^t - \overline{w_s^t}\right)\end{array}\right] \\ & \left[\begin{array}{c}\left(\sum_{i=1}^{m_c} w_{ics}^{t-a} I\left(c_i^{t-a} \geq k\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{id}^{t-a}}{z_d}\right]_+^\alpha - M_s^\alpha\left(X^{t-a};k,Z\right)\right) \\ -M^\alpha\left(X^{t-a};k,Z\right)\left(w_{cs}^{t-a} - \overline{w_s^{t-a}}\right)\end{array}\right]\end{aligned} \tag{39}$$

Finally the asymptotic standard error of $\Delta\%M_a^\alpha$ for the panel-data case is:

$$SE\left(\Delta\%M_a^\alpha\left(X^t, X^{t-a};k,Z\right)\right) = \sqrt{\widehat{var_{\Delta\%M^\alpha}}}. \tag{40}$$

References

Angus Deaton. *The Analysis of Household Surveys. A Microeconometric Approach to Development Policy*. The Johns Hopkins University press, 1997.