Tests of stochastic dominance for multivariate distributions

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- Instead it may be necessary to perform tests on functions of the joint distribution functions!
- Whether we need to check the joint distributions or not depends on the cross derivatives of the class of welfare functions for which we are trying to establish partial orderings
- In other words, it depends on whether the class of functions is characterized by neutrality among the dimensions or not (i.e. they are either substitutes or complements, or both at times)

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   - Declare $A = B$ *if and only if* $A = B$ for all variables $(x_1, ..., x_V)$
   - Declare indeterminacy (i.e. no dominance and no homogeneity) *if and only if* $\exists_{x_i}|ADB$ and $\exists_{x_{j\neq i}}|BDA$

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- There are several conditions but just a few tests (or possible extensions of univariate ones) for these situations
- Some tests are exclusive for continuous variables, others for discrete variables, and others for combinations
- We are going to study two of these for poverty functions: one for continuous variables and one for a combination of continuous and discrete variables (both by Duclos, Sahn and Younger, 2006)

Refreshing dominance conditions and further relevance in multivariate settings

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A stochastic dominance condition is a relationship between functions of the distribution functions of a pair of samples, A and B, which in turn establishes a consistent ranking of A versus B across a range of evaluation functions belong to a specific class This is the form of a typical stochastic dominance condition (the left hand side is by definition the relationship between the functions):

$$R[G_A, G_B] \longleftrightarrow W_A(x) \geqslant W_B(x) \forall x \wedge \exists x | W_A(x) > W_B(x) \forall W \in W^D$$

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- It is a fantastic result.
- It allows for a wide range of conditions.
- The conditions provided allow for poverty comparisons robust not just to dimension weights but chiefly to different poverty lines representing different identification criteria (e.g. union, intersection, intermediate)
- Main drawback: it only considers classes of welfare functions characterized by substitutability between dimensions

The test of Duclos et al. (2006): preliminary notation

The conditions work over additive poverty functions:

$$P(\lambda) = \int\int_{\Lambda(\lambda)} \pi(x, y; \lambda) dF(x, y)$$

The individual poverty function is characterized by:
$\pi(x, y; \lambda) \geq 0$ if $\lambda(x, y) \leq 0$ and $\pi(x, y; \lambda) = 0$ otherwise

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The multiplicative FGT surface (upon which the conditions are derived) is:

$$P^{\alpha_x, \alpha_y}(z_x, z_y) = \int_0^{z_y} \int_0^{z_x} (z_x - x)^{\alpha_x} (z_y - y)^{\alpha_y} dF(x, y)$$

where α_x and α_y are positive integers

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$$\Delta P(\lambda) > 0 \forall P(\lambda) \in \ddot{\prod}^{1,1}(\lambda^*) \longleftrightarrow \Delta P^{0,0}(x,y) > 0 \forall x, y \in \Lambda(\lambda^*)$$

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These conditions can be extended into higher orders of dominance by dimension (the α's) and more dimensions

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3. For instance we can test $H_0 : P_A^{\alpha_x,\alpha_y} \leq P_B^{\alpha_x,\alpha_y}$ against $H_a : P_A^{\alpha_x,\alpha_y} > P_B^{\alpha_x,\alpha_y}$

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4. Rejection of that null hypothesis in favor of that alternative would require that $Z_{\Delta P} \geq Z_{critical} \forall x, y$ and $\exists x, y | Z_{\Delta P} > Z_{critical}$

Testing strategy: the covariance matrices

As in the univariate case we have the case of independent samples (e.g. two countries) and dependent samples (e.g. panel data)

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- Now we need to account for more dimensions
- We are estimating the standard errors of differences of FGT as opposed to differences in functions of probabilities (e.g. the factorial is not considered)

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$$cov[\widehat{\widehat{P}^{\alpha_x,\alpha_y}_{A,t}(z_x, zy), \widehat{P}^{\alpha_x,\alpha_y}_{A,t+1}}(z_x, zy)] =$$

$$(\frac{1}{N})^2 \sum_{i=1}^{N} (z_y - y_{i,t})_+^{\alpha_y} (z_x - x_{i,t})_+^{\alpha_x} (z_y - y_{i,t+1})_+^{\alpha_y} (z_x - x_{i,t+1})_+^{\alpha_x}$$

$$-(\frac{1}{N})\frac{1}{N} \sum_{i=1}^{N} (z_y - y_{i,t})_+^{\alpha_y} (z_x - x_{i,t})_+^{\alpha_x} \frac{1}{N} \sum_{i=1}^{N} (z_y - y_{i,t+1})_+^{\alpha_y} (z_x - x_{i,t+1})_+^{\alpha_x}$$

The test of Duclos et al. (2006) for one continuous and several discrete variables

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- The generic aggregate poverty index is defined over several poverty lines: $P(z(1,1); z(1,2); ...; z(1, K*); ...; z(K, K*))$

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It assumes ordinality in all discrete variables (sensible)

The test of Duclos et al. (2006) for one continuous and several discrete variables: the example they provide

They find a dominance condition for a general class named $\dot{\Pi}^1(z(1,1); z(1,2); ...; z(1, K*); ...; z(K, K*))$

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4. $\pi^{(1)}_{k,k*} \leq \pi^{(1)}_{k,k*+1} \leq 0 \; \forall x, k, k*$
5. $\pi_{k,k*}(z(k,k*)) = 0 \; \forall k = 1, ..., K; k* = 1, ..., K*$

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The condition for this class $\dot{\Pi}^1(z(1,1); z(1,2); ...; z(1,K*); ...; z(K,K*))$ is:

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The condition for this class $\dot{\Pi}^1(z(1,1); z(1,2); ...; z(1,K*); ...; z(K,K*))$ is:

$$\Delta P(\zeta(1,1),...,\zeta(K,K*)) > 0 \forall P(\zeta(1,1),...,\zeta(K,K*)) \in \dot{\Pi}^1$$

$$\wedge \forall \zeta(k,k*) \in [0, z(k,k*)], k = 1,...,K; k* = 1,...,K*$$

$$\longleftrightarrow \sum_{k=1}^{i} \sum_{k*=1}^{j} \Delta P^0(k,k*;\zeta) > 0, \forall \zeta \in [0, z(i,j)]$$

$$\wedge i = 1,...,K; j = 2,...,K*$$

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The procedures are very similar as in the previous cases, for instance:

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$$\sum_{k=1}^{i}\sum_{k*=1}^{j}\widehat{P^{\alpha}}(k,k*;\zeta)=\frac{1}{N}\sum_{m=1}^{N}(\zeta(k_m,k*_m)-x_m)_+^{\alpha}I(k_m\leq i\wedge k*_m\leq j)$$

The test of Duclos et al. (2006) for one continuous and several discrete variables: estimations of statistics

Another example, the covariance of dependent samples with one continuous variable and one discrete variable:

$$cov(\sum_{k=1}^{j} \widehat{P_t^{\alpha}}(k;\zeta), \sum_{k=1}^{j} \widehat{P_{t+1}^{\alpha}}(k;\zeta)) =$$

$$\frac{1}{N^2}\sum_{m=1}^{N}(\zeta(k_{m,t}) - x_{m,t})_+^{\alpha}(\zeta(k_{m,t+1}) - x_{m,t+1})_+^{\alpha} I(k_{m,t} \leq j \wedge k_{m,t+1} \leq j)$$

$$-(\frac{1}{N})\sum_{k=1}^{j} \widehat{P_t^{\alpha}}(k;\zeta)\sum_{k=1}^{j} \widehat{P_{t+1}^{\alpha}}(k;\zeta)$$