Tests of stochastic dominance for multivariate distributions

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- In multivariate settings comparing the marginal distributions of A and B for every variable may not be appropriate to ascertain stochastic dominance (and partial orderings)
- Instead it may be necessary to perform tests on functions of the joint distribution functions!
- Whether we need to check the joint distributions or not depends on the cross derivatives of the class of welfare functions for which we are trying to establish partial orderings
- In other words, it depends on whether the class of functions is characterized by neutrality among the dimensions or not (i.e. they are either substitutes or complements, or both at times)

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   - Similarly for B D A
   - Declare $A = B$ *if and only if* $A = B$ for all variables $(x_1, ..., x_V)$
   - Declare indeterminacy (i.e. no dominance and no homogeneity) *if and only if* $\exists_{x_i} | ADB$ and $\exists_{x_{j \neq i}} | BDA$

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- There are several conditions but just a few tests (or possible extensions of univariate ones) for these situations
- Some tests are exclusive for continuous variables, others for discrete variables, and others for combinations
- We are going to study two of these for poverty functions: one for continuous variables and one for a combination of continuous and discrete variables (both by Duclos, Sahn and Younger, 2006)

Refreshing dominance conditions and further relevance in multivariate settings

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A stochastic dominance condition is a relationship between functions of the distribution functions of a pair of samples, A and B, which in turn establishes a consistent ranking of A versus B across a range of evaluation functions belong to a specific class This is the form of a typical stochastic dominance condition (the left hand side is by definition the relationship between the functions):

$$R[G_A, G_B] \longleftrightarrow W_A(x) \geqslant W_B(x) \forall x \wedge \exists x | W_A(x) > W_B(x) \forall W \in W^D$$

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- It is a fantastic result
- It allows for a wide range of conditions
- The conditions provided allow for poverty comparisons robust not just to dimension weights but chiefly to different poverty lines representing different identification criteria (e.g. union, intersection, intermediate)
- Main drawback: it only considers classes of welfare functions characterized by substitutability between dimensions

The test of Duclos et al. (2006): preliminary notation

The conditions work over additive poverty functions:

$$P(\lambda) = \int\int_{\Lambda(\lambda)} \pi(x, y; \lambda) dF(x, y)$$

The individual poverty function is characterized by:
$\pi(x, y; \lambda) \geq 0$ if $\lambda(x, y) \leq 0$ and $\pi(x, y; \lambda) = 0$ otherwise

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The multiplicative FGT surface (upon which the conditions are derived) is:

$$P^{\alpha_x, \alpha_y}(z_x, z_y) = \int_0^{z_y} \int_0^{z_x} (z_x - x)^{\alpha_x} (z_y - y)^{\alpha_y} dF(x, y)$$

where α_x and α_y are positive integers

The test of Duclos et al. (2006): examples of classes for two continuous variables

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$$\Delta P(\lambda) > 0 \forall P(\lambda) \in \ddot{\prod}^{1,1}(\lambda^*) \longleftrightarrow \Delta P^{0,0}(x,y) > 0 \forall x, y \in \Lambda(\lambda^*)$$

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These conditions can be extended into higher orders of dominance by dimension (the α's) and more dimensions

Testing strategy

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3. For instance we can test $H_0 : P_A^{\alpha_x,\alpha_y} \leq P_B^{\alpha_x,\alpha_y}$ against $H_0 : P_A^{\alpha_x,\alpha_y} > P_B^{\alpha_x,\alpha_y}$

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4. Rejection of that null hypothesis in favor of that alternative would require that $Z_{\Delta P} \geq Z_{critical} \forall x, y$ and $\exists x, y | Z_{\Delta P} > Z_{critical}$

Testing strategy: the covariance matrices

As in the univariate case we have the case of independent samples (e.g. two countries) and dependent samples (e.g. panel data)

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- Now we need to account for more dimensions
- We are estimating the standard errors of differences of FGT as opposed to differences in functions of probabilities (e.g. the factorial is not considered)

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$$cov[\widehat{P}^{\alpha_x,\alpha_y}_{A,t}\widehat{(z_x, zy), \widehat{P}}^{\alpha_x,\alpha_y}_{A,t+1}](z_x, zy) =$$

$$(\frac{1}{N})^2 \sum_{i=1}^{N} (z_y - y_{i,t})_+^{\alpha_y} (z_x - x_{i,t})_+^{\alpha_x} (z_y - y_{i,t+1})_+^{\alpha_y} (z_x - x_{i,t+1})_+^{\alpha_x}$$

$$-(\frac{1}{N}) \frac{1}{N} \sum_{i=1}^{N} (z_y - y_{i,t})_+^{\alpha_y} (z_x - x_{i,t})_+^{\alpha_x} \frac{1}{N} \sum_{i=1}^{N} (z_y - y_{i,t+1})_+^{\alpha_y} (z_x - x_{i,t+1})_+^{\alpha_x}$$

The test of Duclos et al. (2006) for one continuous and several discrete variables

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- The generic aggregate poverty index is defined over several poverty lines: $P(z(1,1); z(1,2); ...; z(1, K*); ...; z(K, K*)$

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This test seems to be even more relevant in empirical applications

It assumes ordinality in all discrete variables (sensible)

The test of Duclos et al. (2006) for one continuous and several discrete variables: the example they provide

They find a dominance condition for a general class named $\dot{\Pi}^1(z(1,1); z(1,2); ...; z(1, K*); ...; z(K, K*))$

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4. $\pi^{(1)}_{k,k*} \leq \pi^{(1)}_{k,k*+1}\ \forall x, k, k*$
5. $\pi_{k,k*}(z(k,k*)) = 0\ \forall k = 1, ..., K; k* = 1, ..., K*$

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The condition for this class $\dot{\Pi}^1(z(1,1); z(1,2); ...; z(1,K*); ...; z(K,K*)$ is:

$$\Delta P(\zeta(1,1),...,\zeta(K,K*)) > 0 \forall P(\zeta(1,1),...,\zeta(K,K*)) \in \dot{\Pi}^1$$

$$\wedge \forall \zeta(k,k*) \in [0, z(k,k*)], k = 1,...,K; k* = 1,...,K*$$

$$\longleftrightarrow \sum_{k=1}^{i} \sum_{k*=1}^{j} \Delta P^0(k,k*;\zeta) > 0, \forall \zeta \in [0, z(i,j)]$$

$$\wedge i = 1,...,K; j = 2,...,K*$$

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$$\sum_{k=1}^{i}\sum_{k*=1}^{j}\widehat{P^{\alpha}}(k,k*;\zeta)=\frac{1}{N}\sum_{m=1}^{N}(\zeta(k_m,k*_m)-x_m)_+^{\alpha}I(k_m\leq i\wedge k*_m\leq j)$$

The test of Duclos et al. (2006) for one continuous and several discrete variables: estimations of statistics

Another example, the covariance of dependent samples with one continuous variable and one discrete variable:

$$cov(\sum_{k=1}^{j} \widehat{P_t^\alpha}(k;\zeta), \sum_{k=1}^{j} \widehat{P_{t+1}^\alpha}(k;\zeta)) =$$

$$\frac{1}{N^2} \sum_{m=1}^{N} (\zeta(k_{m,t}) - x_{m,t})_+^\alpha (\zeta(k_{m,t+1}) - x_{m,t+1})_+^\alpha I(k_{m,t+1} \leq j \wedge k_{m,t+1} \leq j$$

$$-(\frac{1}{N}) \sum_{k=1}^{j} \widehat{P_t^\alpha}(k;\zeta) \sum_{k=1}^{j} \widehat{P_{t+1}^\alpha}(k;\zeta)$$