Multidimensional Inequality

Suman Seth

Vanderbilt University & OPHI

3rd September, 2009
Study of inequality is important on its own
Study of inequality is important on its own
Multiple dimensions vs single-dimension
Study of inequality is important on its own

Multiple dimensions vs single-dimension

- Cability approach
Measurement of Multidimensional Inequality

- Study of inequality is important on its own
- Multiple dimensions vs single-dimension
  - Capability approach
  - Basic needs approach
Outline of Today’s Lecture

- Basic framework for today’s discussion
Outline of Today’s Lecture

- Basic framework for today’s discussion
- Related axioms
Outline of Today’s Lecture

- Basic framework for today’s discussion
- Related axioms
- Introduce various inequality indices
Outline of Today’s Lecture

- Basic framework for today’s discussion
- Related axioms
- Introduce various inequality indices
  - Pros and cons of these indices
Basic Theoretical Framework

- $N$ persons and $D$ dimensions

$$\text{Normalized achievement vector: } X = x_{11} x_{12} \cdots x_{ND}$$

$x_{nd}$ is the achievement of the $n$th person in the $d$th dimension

Denote:

$$x_n = (x_{n1}, \ldots, x_{nD})$$

$$x_d = (x_{1d}, \ldots, x_{Nd})$$
Basic Theoretical Framework

- \( N \) persons and \( D \) dimensions

- Normalized achievement vector: \( X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{ND} \end{bmatrix} \)
Basic Theoretical Framework

- **N** persons and **D** dimensions

- Normalized achievement vector: \( X = \begin{bmatrix}
  x_{11} & x_{12} & \cdots & x_{1D} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{N1} & x_{N2} & \cdots & x_{ND}
\end{bmatrix} \)

- \( x_{nd} \) is the achievement of the \( n^{\text{th}} \) person in the \( d^{\text{th}} \) dimension
Basic Theoretical Framework

- $N$ persons and $D$ dimensions
- Normalized achievement vector: $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{ND} \end{bmatrix}$
- $x_{nd}$ is the achievement of the $n^{\text{th}}$ person in the $d^{\text{th}}$ dimension
- $x_{nd} > 0 \ \forall n, d$
- \( N \) persons and \( D \) dimensions

- Normalized achievement vector: \( X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{ND} \end{bmatrix} \)

- \( x_{nd} \) is the achievement of the \( n^{th} \) person in the \( d^{th} \) dimension

- \( x_{nd} > 0 \ \forall \ n, d \)

- Denote: \( x_{n*} = (x_{n1}, ..., x_{nD}) \ \forall \ n \) and \( x_{*d} = (x_{1d}, ..., x_{Nd}) \ \forall \ d \)
Basic Theoretical Framework

- $N$ persons and $D$ dimensions

- Normalized achievement vector: $X = \begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1D} \\
\vdots & \vdots & \ddots & \vdots \\
x_{N1} & x_{N2} & \cdots & x_{ND}
\end{bmatrix}$

- $x_{nd}$ is the achievement of the $n^{th}$ person in the $d^{th}$ dimension

- $x_{nd} > 0 \ \forall n, d$

- Denote: $x_{n^*} = (x_{n1}, \ldots, x_{nD})$ $\forall n$ and $x_{d^*} = (x_{1d}, \ldots, x_{Nd})$ $\forall d$
  - $x_{n^*}$ is the achievement vector of the $n^{th}$ person
$N$ persons and $D$ dimensions

Normalized achievement vector: $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1D} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{ND} \end{bmatrix}$

$x_{nd}$ is the achievement of the $n^{th}$ person in the $d^{th}$ dimension

$x_{nd} > 0 \ \forall n, d$

Denote: $x_{n*} = (x_{n1}, \ldots, x_{nD}) \ \forall n$ and $x_{*d} = (x_{1d}, \ldots, x_{Nd}) \ \forall d$

- $x_{n*}$ is the achievement vector of the $n^{th}$ person
- $x_{*d}$ is the achievement vector of the $d^{th}$ dimension
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$. 
- **Anonymity (ANON).** Personal identity does not matter.
- **Scale Invariance (SINV).** If all elements in $X$ is changed by an equal proportional amount, then inequality does not change $I(\delta X) = I(X)$ where $\delta > 0$.
- **Translation Invariance (TINV):** If all elements in $X$ is increased by an equal additional amount, then inequality does not change $I(X + \lambda) = I(X)$ where $\lambda > 0$. 

Suman Seth (Vanderbilt University & OPHI) Multidim Inequality 3rd September, 2009 5 / 17
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- Normalization (NORM). If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = x$ for all $n$, then $I(X) = 0$
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$.
  - If $x_n = x$ for all $n$, then $I(X) = 0$.

- **Anonymity (ANON).** Personal identity does not matter.
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = x$ for all $n$, then $I(X) = 0$
- **Anonymity (ANON).** Personal identity does not matter
  - **Mathematical definition?**

Mathematical definition?

$I(PX) = I(X)$, where $P$ is an $N \times N$ permutation matrix

Scale Invariance (SINV). If all elements in $X$ is changed by an equal proportional amount, then inequality does not change

$I(\delta X) = I(X)$ where $\delta > 0$

Translation Invariance (TINV): If all elements in $X$ is increased by an equal additional amount, then inequality does not change

$I(X + \lambda) = I(X)$ where $\lambda > 0$
Basic Axioms Satisfied by Inequality Indices: \( I(X) \)

- **Normalization (NORM).** If each person has the same achievement vector, then \( I(X) = 0 \)
  
  If \( x_n = x \) for all \( n \), then \( I(X) = 0 \)

- **Anonymity (ANON).** Personal identity does not matter
  
  Mathematical definition?

  \[ I(PX) = I(X) \], where \( P \) is an \( N \times N \) permutation matrix
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- Normalization (NORM). If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = x$ for all $n$, then $I(X) = 0$
- Anonymity (ANON). Personal identity does not matter
  - Mathematical definition?
  - $I(PX) = I(X)$, where $P$ is an $N \times N$ permutation matrix
- Scale Invariance (SINV). If all elements in $X$ is changed by an equal proportional amount, then inequality does not change
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = x$ for all $n$, then $I(X) = 0$

- **Anonymity (ANON).** Personal identity does not matter
  - Mathematical definition?
  - $I(PX) = I(X)$, where $P$ is an $N \times N$ permutation matrix

- **Scale Invariance (SINV).** If all elements in $X$ is changed by an equal proportional amount, then inequality does not change
  - $I(\delta X) = I(X)$ where $\delta > 0$
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = \mathbf{x}$ for all $n$, then $I(X) = 0$

- **Anonymity (ANON).** Personal identity does not matter
  - Mathematical definition?
  - $I(PX) = I(X)$, where $P$ is an $N \times N$ permutation matrix

- **Scale Invariance (SINV).** If all elements in $X$ is changed by an equal proportional amount, then inequality does not change
  - $I(\delta X) = I(X)$ where $\delta > 0$

- **Translation Invariance (TINV):** If all elements in $X$ is increased by an equal additional amount, then inequality does not change
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Normalization (NORM).** If each person has the same achievement vector, then $I(X) = 0$
  - If $x_n = x$ for all $n$, then $I(X) = 0$

- **Anonymity (ANON).** Personal identity does not matter
  - Mathematical definition?
  - $I(PX) = I(X)$, where $P$ is an $N \times N$ permutation matrix

- **Scale Invariance (SINV).** If all elements in $X$ is changed by an equal proportional amount, then inequality does not change
  - $I(\delta X) = I(X)$ where $\delta > 0$

- **Translation Invariance (TINV):** If all elements in $X$ is increased by an equal additional amount, then inequality does not change
  - $I(X + \lambda) = I(X)$ where $\lambda > 0$
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- Population Replication Invariance (POPRI). Replication of the same population several times does not change overall inequality.
Basic Axioms Satisfied by Inequality Indices: I(X)

- Population Replication Invariance (POPRI). Replication of the same population several times does not change overall inequality.
- Decomposability (DECOM). Overall inequality can be expressed as a general function of the subgroup means, population sizes and inequality values.
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Population Replication Invariance (POPRI).** Replication of the same population several times does not change overall inequality.

- **Decomposability (DECOM).** Overall inequality can be expressed as a general function of the subgroup means, population sizes and inequality values.

  For two groups of size $N_1$ and $N_2$ such that $N_1 + N_2 = N$, $I(X) = f\left(I(X_{N_1}), I(X_{N_2}), \bar{X}_{N_1}, \bar{X}_{N_2}, N_1, N_2\right)$, where $\bar{X}_{N_1}$ and $\bar{X}_{N_2}$ are the mean vectors of $X_{N_1}$ and $X_{N_2}$.
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- **Population Replication Invariance (POPRI).** Replication of the same population several times does not change overall inequality.

- **Decomposability (DECOM).** Overall inequality can be expressed as a general function of the subgroup means, population sizes and inequality values.
  
  For two groups of size $N_1$ and $N_2$ such that $N_1 + N_2 = N$, $I(X) = f \left( I(X_{N_1}), I(X_{N_2}), \bar{X}_{N_1}, \bar{X}_{N_2}, N_1, N_2 \right)$, where $\bar{X}_{N_1}$ and $\bar{X}_{N_2}$ are the mean vectors of $X_{N_1}$ and $X_{N_2}$.

- **Subgroup Consistency (SUBCON).** If the inequality of one subgroup rises and the other is unaltered, then overall inequality rises.
Basic Axioms Satisfied by Inequality Indices: $I(X)$

- Population Replication Invariance (POPRI). Replication of the same population several times does not change overall inequality.

- Decomposability (DECOM). Overall inequality can be expressed as a general function of the subgroup means, population sizes and inequality values.
  
  For two groups of size $N_1$ and $N_2$ such that $N_1 + N_2 = N$, $I(X) = f \left( I(X_{N_1}), I(X_{N_2}), \bar{X}_{N_1}, \bar{X}_{N_2}, N_1, N_2 \right)$, where $\bar{X}_{N_1}$ and $\bar{X}_{N_2}$ are the mean vectors of $X_{N_1}$ and $X_{N_2}$.

- Subgroup Consistency (SUBCON). If the inequality of one subgroup rises and the other is unaltered, then overall inequality rise.

- Continuity (CONTN). $I(H)$ does not change abruptly due to a change in any of the elements in $H$. 
Uniform Majorization (UM). If \( X \) is multiplied by a bistochastic matrix (B) rendering the distributions of attributes less spread out, then \( I(X) > I(XB) \).
Axioms Sensitive to Inequality Across Persons

- **Uniform Majorization (UM)**. If $X$ is multiplied by a bistochastic matrix ($B$) rendering the distributions of attributes less spread out, then $I(X) > I(XB)$.

Example: $X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix}$, $\bar{X} = \begin{bmatrix} 0.35 & 0.35 & 0.6 \\ 0.35 & 0.35 & 0.6 \end{bmatrix}$
Uniform Majorization (UM). If $X$ is multiplied by a bistochastic matrix (B) rendering the distributions of attributes less spread out, then $I(X) > I(XB)$.

Example: $X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix}$, $\bar{X} = \begin{bmatrix} 0.35 & 0.35 & 0.6 \\ 0.35 & 0.35 & 0.6 \end{bmatrix}$

Correlation Increasing Transfer (CIT). $Y$ is derived from $X$ by a correlation increasing transfer if, for some rows $n'$ and $n''$, $n' < n''$, $y_{n'} = x_{n'} \land x_{n''}$, $y_{n''} = x_{n'} \lor x_{n''}$, and $y_n = x_n$ for all $n \notin \{n', n''\}$, where $x \land y = (\min\{x_{n'1}, y_{n'1}\}, \ldots, \min\{x_{n''1}, y_{n''1}\})$ and $x \lor y = (\max\{x_{n'1}, y_{n'1}\}, \ldots, \max\{x_{n''1}, y_{n''1}\})$ (Boland and Proschan 1988)
Example of CIT (Y is obtained from X by a CIT)
Example of CIT (Y is obtained from X by a CIT)

\[
X = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.3 & 0.8 \\
0.3 & 0.4 & 0.4
\end{bmatrix}, \quad Y = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.4 & 0.8 \\
0.3 & 0.3 & 0.4
\end{bmatrix}
\]
Example of CIT (Y is obtained from X by a CIT)

\[
X = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.3 & 0.8 \\
0.3 & 0.4 & 0.4
\end{bmatrix}, \quad Y = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.4 & 0.8 \\
0.3 & 0.3 & 0.4
\end{bmatrix}
\]

Correlation Increasing Majorization (CIM). Higher correlation between attributes, for given marginal distributions, should lead to more inter-personal inequality, that is, \( I(Y) > I(X) \)
Example of CIT (Y is obtained from X by a CIT)

\[
X = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.3 & 0.8 \\
0.3 & 0.4 & 0.4
\end{bmatrix}, \quad Y = \begin{bmatrix}
0.8 & 0.8 & 0.3 \\
0.4 & 0.4 & 0.8 \\
0.3 & 0.3 & 0.4
\end{bmatrix}
\]

Correlation Increasing Majorization (CIM). Higher correlation between attributes, for given marginal distributions, should lead to more inter-personal inequality, that is, \( I(Y) > I(X) \)

Note - For indices with two stage aggregation approach, if the aggregation takes place first across persons and then across dimensions, indices do not satisfy CIT
An absolute inequality index must satisfy translation invariance axiom besides other essential axioms.
Absolute vs. relative inequality index

- An *absolute inequality index* must satisfy *translation invariance axiom* besides other essential axioms

- A *relative inequality index* must satisfy *scale invariance axiom* besides other essential axioms
Proposed Multidimensional Inequality Indices

- Bourguignon Index (1999)
Proposed Multidimensional Inequality Indices

- Bourguignon Index (1999)
- Maasoumi Index (1986, 1999)
Proposed Multidimensional Inequality Indices

- Bourguignon Index (1999)
- Maasoumi Index (1986, 1999)
- Tsui Index (1995, 1999)
Proposed Multidimensional Inequality Indices

- Bourguignon Index (1999)
- Maasoumi Index (1986, 1999)
- Tsui Index (1995, 1999)
- Gajdos and Weymark Index (2005)
Proposed Multidimensional Inequality Indices

- Bourguignon Index (1999)
- Maasoumi Index (1986, 1999)
- Tsui Index (1995, 1999)
- Gajdos and Weymark Index (2005)
- Decanq and Lugo (2008)
First, derives a well-being index
First, derives a well-being index

First Stage: Aggregates across dimensions by the aggregator function

\[ U_n = \left[ \mu_\beta (x_{n1}, ..., x_{nD}) \right]^\alpha \]; \( \beta < 1 \), \( 0 < \alpha < 1 \).
Bourguignon Index (1999)

- First, derives a well-being index
- First Stage: Aggregates across dimensions by the aggregator function
  \[ U_n = \left( \mu_\beta(x_{n1}, \ldots, x_{nD}) \right)^\alpha; \beta < 1, 0 < \alpha < 1. \]
- Second stage: Aggregates across persons by the aggregator function:
  \[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]
First, derives a well-being index

First Stage: Aggregates across dimensions by the aggregator function
\[ U_n = \left[ \mu_\beta (x_{n1}, ..., x_{nD}) \right]^\alpha ; \beta < 1, 0 < \alpha < 1. \]

Second stage: Aggregates across persons by the aggregator function:
\[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]

Defines \( \bar{W} = \bar{U} \), where \( \bar{U} = \left[ \mu_\beta (\mu_1 (x_{11}), ..., \mu_1 (x_{1D})) \right]^\alpha \)
Bourguignon Index (1999)

- First, derives a well-being index
- First Stage: Aggregates across dimensions by the aggregator function
  \[ U_n = \left[ \mu_\beta (x_{n1}, \ldots, x_{nD}) \right]^\alpha; \beta < 1, 0 < \alpha < 1. \]
- Second stage: Aggregates across persons by the aggregator function:
  \[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]
- Defines \( \bar{W} = \bar{U} \), where \( \bar{U} = \left[ \mu_\beta (\mu_1 (x_{*1}), \ldots, \mu_1 (x_{*D})) \right]^\alpha \)
- Inequality index
  \[ I_B = 1 - \frac{W}{\bar{W}} \]
Bourguignon Index (1999)

- First, derives a well-being index

First Stage: Aggregates across dimensions by the aggregator function
\[ U_n = \left[ \mu_{\beta} \left( x_{n1}, ..., x_{nD} \right) \right]^\alpha ; \beta < 1, \ 0 < \alpha < 1. \]

Second stage: Aggregates across persons by the aggregator function:
\[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]

Defines \( \bar{W} = \bar{U} \), where \( \bar{U} = \left[ \mu_{\beta} \left( \mu_{1} \left( x_{*1} \right), ..., \mu_{1} \left( x_{*D} \right) \right) \right]^\alpha \)

- Inequality index
\[ I_B = 1 - \frac{W}{\bar{W}} \]

- \( \beta \) is substitution parameter and \( \alpha \) is inequality aversion parameter
Bourguignon Index (1999)

- First, derives a well-being index

  **First Stage:** Aggregates across dimensions by the aggregator function

  \[ U_n = \left[ \mu_\beta (x_{n1}, ..., x_{nD}) \right]^\alpha ; \beta < 1, \ 0 < \alpha < 1. \]

- Second stage: Aggregates across persons by the aggregator function:

  \[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]

- Defines \( \bar{W} = \bar{U} \), where \( \bar{U} = \left[ \mu_\beta (\mu_1 (x_{*1}), ..., \mu_1 (x_{*D})) \right]^\alpha \)

- Inequality index

  \[ I_B = 1 - \frac{W}{\bar{W}} \]

- \( \beta \) is substitution parameter and \( \alpha \) is inequality aversion parameter

- \( I_B \) satisfies NM, SP, SI, D, RI, and both forms of inequality sensitive axioms
First, derives a well-being index

First Stage: Aggregates across dimensions by the aggregator function

\[ U_n = \left[ \mu_\beta (x_{n1}, \ldots, x_{nD}) \right]^\alpha ; \beta < 1, \ 0 < \alpha < 1. \]

Second stage: Aggregates across persons by the aggregator function:

\[ W = \frac{1}{N} \sum_{i=1}^{N} U_n \]

Defines \( \ddot{W} = \ddot{U} \), where \( \ddot{U} = \left[ \mu_\beta (\mu_1 (x_{*1}), \ldots, \mu_1 (x_{*D})) \right]^\alpha \)

Inequality index

\[ I_B = 1 - \frac{W}{\ddot{W}} \]

\( \beta \) is substitution parameter and \( \alpha \) is inequality aversion parameter

\( I_B \) satisfies NM, SP, SI, D, RI, and both forms of inequality sensitive axioms

Inequality increases with correlation when \( \alpha < \beta \)
Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)
Bourguignon Index (1999)

- Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

- First stage aggregation across dimensions yields

\[
U_1 = 0.68, \quad U_2 = 0.63, \quad U_3 = 0.60.
\]
Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

First stage aggregation across dimensions yields

\[ U_1 = 0.68, \ U_2 = 0.63, \ U_3 = 0.60. \]

Second stage aggregation across persons yields

\[ W = \frac{1}{3} \left( 0.68 + 0.63 + 0.60 \right) = 0.64 \]
Bourguignon Index (1999)

- Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

- First stage aggregation across dimensions yields
  \[
  U_1 = 0.68, \quad U_2 = 0.63, \quad U_3 = 0.60.
  \]

- Second stage aggregation across persons yields
  \[
  W = \frac{1}{3} (0.68 + 0.63 + 0.60) = 0.64
  \]

- Create \( h = (0.5, 0.5, 0.5) \)
Bourguignon Index (1999)

Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix}, \beta = -2, \alpha = 0.5, a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

First stage aggregation across dimensions yields

\[ U_1 = 0.68, \quad U_2 = 0.63, \quad U_3 = 0.60. \]

Second stage aggregation across persons yields

\[ W = \frac{1}{3} (0.68 + 0.63 + 0.60) = 0.64 \]

Create \( h = (0.5, 0.5, 0.5) \)

Then \( \bar{U} = \left[ \mu_{-2} (0.5, 0.5, 0.5) \right]^{0.5} = 0.71. \quad \bar{W} = 0.71 \)
Bourguignon Index (1999)

- Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

- First stage aggregation across dimensions yields

\[ U_1 = 0.68, \ U_2 = 0.63, \ U_3 = 0.60. \]

- Second stage aggregation across persons yields

\[ W = \frac{1}{3} (0.68 + 0.63 + 0.60) = 0.64 \]

- Create \( h = (0.5, 0.5, 0.5) \)

- Then

\[ \bar{U} = \left[ \mu_{-2} (0.5, 0.5, 0.5) \right]^{0.5} = 0.71. \ \bar{W} = 0.71 \]

- Inequality index

\[ I_B = 1 - \frac{0.64}{0.71} = 0.099 \]
Bourguignon Index (1999)

- Example: \( X = \begin{bmatrix} 0.8 & 0.8 & 0.3 \\ 0.4 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.4 \end{bmatrix} \), \( \beta = -2 \), \( \alpha = 0.5 \), \( a = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \)

- First stage aggregation across dimensions yields
  \[ U_1 = 0.68, \ U_2 = 0.63, \ U_3 = 0.60. \]

- Second stage aggregation across persons yields
  \[ W = \frac{1}{3} (0.68 + 0.63 + 0.60) = 0.64 \]

- Create \( h = (0.5, 0.5, 0.5) \)

- Then \( \bar{U} = \left[ \mu_{-2} (0.5, 0.5, 0.5) \right]^{0.5} = 0.71 \). \( \bar{W} = 0.71 \)

- Inequality index
  \[ I_B = 1 - \frac{0.64}{0.71} = 0.099 \]

- Problems: role of inequality aversion parameter is not clear
Maasoumi Index (1986, 1999)

- A two stage procedure

\[ \bar{S} = \frac{1}{N} \sum_{i=1}^{N} U_n \]
Maasoumi Index (1986, 1999)

- A two stage procedure
- The first stage is a generalized mean

\[ U_n = \mu_\beta (x_{n1}, \ldots, x_{nD}). \]
Maasoumi Index (1986, 1999)

- A two stage procedure
- The first stage is a generalized mean
  \[ U_n = \mu_\beta (x_{n1}, \ldots, x_{nD}) \].
- The second stage is a generalized entropy
  \[
  I_M = \begin{cases}
  \frac{1}{\alpha(1-\alpha)} \frac{1}{N} \sum_{i=1}^{n} \left( 1 - \left( \frac{U_n}{\bar{S}} \right)^\alpha \right) & \text{for } \alpha \neq 0, 1. \\
  \frac{1}{N} \sum_{i=1}^{n} \log \left( \frac{\bar{S}}{U_n} \right) & \text{for } \alpha = 0 \\
  \frac{1}{N} \sum_{i=1}^{n} \frac{U_n}{\bar{S}} \log \left( \frac{U_n}{\bar{S}} \right) & \text{for } \alpha = 1
  \end{cases}
  \]
Maasoumi Index (1986, 1999)

- A two stage procedure
- The first stage is a generalized mean
  \[ U_n = \mu_\beta (x_{n1}, \ldots, x_{nD}) \].
- The second stage is a generalized entropy
  \[
  I_M = \begin{cases} 
  \frac{1}{\alpha(1-\alpha)} \frac{1}{N} \sum_{i=1}^{n} \left(1 - \left(\frac{U_n}{\bar{S}}\right)^\alpha\right) & \text{for } \alpha \neq 0, 1. \\
  \frac{1}{N} \sum_{i=1}^{n} \log \left(\frac{\bar{S}}{U_n}\right) & \text{for } \alpha = 0 \\
  \frac{1}{N} \sum_{i=1}^{n} \frac{U_n}{\bar{S}} \log \left(\frac{U_n}{\bar{S}}\right) & \text{for } \alpha = 1 
  \end{cases}
  \]
- \( \bar{S} = \frac{1}{N} \sum_{i=1}^{N} U_n \)
Maasoumi Index (1986, 1999)

- A two stage procedure
- The first stage is a generalized mean
  \[ U_n = \mu_\beta (x_{n1}, ..., x_{nD}) \]
- The second stage is a generalized entropy
  \[ I_M = \begin{cases} \frac{1}{\alpha(1-\alpha)} \frac{1}{N} \sum_{i=1}^{n} \left( 1 - \left( \frac{U_n}{\bar{S}} \right)^{\alpha} \right) & \text{for } \alpha \neq 0, 1. \\ \frac{1}{N} \sum_{i=1}^{n} \log \left( \frac{\bar{S}}{U_n} \right) & \text{for } \alpha = 0 \\ \frac{1}{N} \sum_{i=1}^{n} \frac{U_n}{\bar{S}} \log \left( \frac{U_n}{\bar{S}} \right) & \text{for } \alpha = 1 \end{cases} \]

- \[ \bar{S} = \frac{1}{N} \sum_{i=1}^{N} U_n \]
- Problems: Not sure what restrictions on parameter satisfies different transfer properties
Tsui Index (1995, 1999)

- Tsui (1995)

$$I_{TRI} = 1 - \left[ \frac{1}{N} \sum_{n=1}^{N} \left( \prod_{d=1}^{D} \left( \frac{x_{nd}}{\mu_d} \right)^{a_d} \right) \right]^{1/\sum_{i=1}^{D} a_d}$$

Tsui also developed more indices in 1999 based on generalized entropy. Unlike Maasoumi, these indices had parameter specification to satisfy transfer.

Problem: Tsui parameters are not interpretable.
Tsui Index (1995, 1999)

- Tsui (1995)

\[ I_{TRI} = 1 - \left[ \frac{1}{N} \sum_{n=1}^{N} \left( \prod_{d=1}^{D} \left( \frac{x_{nd}}{\mu_d} \right)^{a_d} \right) \right]^{1/ \sum_{d=1}^{D} a_d} \]

- Tsui also developed more indices in 1999 based on generalized entropy
Tsui Index (1995, 1999)

- Tsui (1995)
  \[
  I_{TRI} = 1 - \left[ \frac{1}{N} \sum_{n=1}^{N} \left( \prod_{d=1}^{D} \left( \frac{x_{nd}}{\mu_d} \right)^{a_d} \right) \right]^{1/\sum_{d=1}^{D} a_d}
  \]

- Tsui also developed more indices in 1999 based on generalized entropy
- Unlike Maasoumi, these indices had parameter specification to satisfy transfer.

Suman Seth (Vanderbilt University & OPHI)
Tsui Index (1995, 1999)

- Tsui (1995)

\[ I_{TRI} = 1 - \left[ \frac{1}{N} \sum_{n=1}^{N} \left( \prod_{d=1}^{D} \left( \frac{x_{nd}}{\mu_d} \right)^{a_d} \right) \right]^{1/\sum_{i=1}^{D} a_d} \]

- Tsui also developed more indices in 1999 based on generalized entropy
- Unlike Maasoumi, these indices had parameter specification to satisfy transfer.
- Problem: Tsui parameters are not interpretable.
Gajdos and Weymark Index (2005)
Multidimensional Generalized Gini Indices

- Gajdos and Weymark Index (2005)
  - First stage: Gini social evaluation function, which is the generalized Gini index.
  - Second stage: generalized mean across dimensions.

\[ I_{GW} = \mu^{\beta}(U_1, \ldots, U_D) \] for \( \beta \neq 1 \)

Limitation: the order of aggregation makes \( I_{GW} \) to be not strictly sensitive to correlation among dimensions.
Gajdos and Weymark Index (2005)

- First stage: Gini social evaluation function, which is the generalized Gini index.
- Second stage: generalized mean across dimensions.

\[ I_{GW} = \mu_\beta (U_1, ..., U_D) \text{ for } \beta \leq 1 \]
Gajdos and Weymark Index (2005)

- First stage: Gini social evaluation function, which is the generalized Gini index.
- Second stage: generalized mean across dimensions.

\[ I_{GW} = \mu_{\beta}(U_1, ..., U_D) \text{ for } \beta \leq 1 \]

Limitation: the order of aggregation makes \( I_{GW} \) to be not strictly sensitive to correlation among dimensions.
Decancq and Lugo (2008)
• Decancq and Lugo (2008)
• Reversed order of aggregation
Decancq and Lugo (2008)

Reversed order of aggregation

First stage: generalized mean across dimensions.

\[ U_n = \mu_\beta(x_{n*}) \quad \text{for} \quad \beta \leq 1 \]
Decancq and Lugo (2008)

Reversed order of aggregation

First stage: generalized mean across dimensions.

\[ U_n = \mu_\beta(x_{n*}) \text{ for } \beta \leq 1 \]

Second stage: Gini social evaluation function, which is generalized Gini index.
Summary

- Introduced axioms
• Introduced axioms
• Introduced the basic frameworks
Summary

- Introduced axioms
- Introduced the basic frameworks
- Discussed various multidimensional inequality indices
Summary

- Introduced axioms
- Introduced the basic frameworks
- Discussed various multidimensional inequality indices
- Discussed Pros and Cons of these indices