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Conditions for the Most Robust Poverty Comparisons Using the Alkire-Foster Family of Measures

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Abstract
In the burgeoning literature on multidimensional poverty indices, the Alkire-Foster (AF) measures stand out for their resilience in identifying the multidimensionally poor with cut-off criteria covering the spectrum from the union approach to the intersection approach. The intuitiveness and easy applicability of the measures' identificationand aggregation methods are reflected in the increasing use of the AF measures in poverty measurement, as well as in other fields. This paper extends the dominance results derived by Lasso de la Vega (2009) and Alkire and Foster (2010) for the adjusted headcount ratio and develops a new condition whose fulfillment ensures the robustness of comparisons using the adjusted headcount ratio for any choice of multidimensional cut-off and for any weights and poverty lines. The paper then derives a first-order dominance condition for the whole Alkire-Foster family (that is, for continuous variables).

Keywords: Multidimensional poverty; stochastic dominance.

JEL classification: I32

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Introduction

The case for an assessment of poverty considering multiple deprivations has been well argued for a long time.[1] While there is a broad consensus about the multidimensionality of poverty, there is a debate as to whether the multiple indicators of deprivations should be brought together into a composite index or not.[2] On the other hand, it seems that a composite measure of multiple deprivations is unavoidable when the purpose is to quantify the incidence of multiple deprivations *within the same individuals.* In practice, one of the approaches proposed to measure multidimensional poverty with a composite index is the counting approach, which is based on counting the number of dimensions in which people are deprived.[3] The approach has gained recent popularity with the Alkire-Foster (AF) family of poverty indices (Alkire and Foster, 2010). These indices identify the multidimensionally poor by counting the number of dimensions in which they are deprived. First, deprivation in any particular dimension is determined by comparing the achievement in that dimension against the respective dimension-specific poverty line. This is done for all dimensions/variables and then the (weighted) number of deprivations is compared against a multidimensional-deprivation cut-off.[4] By changing the cut-off from some minimum value up to the total number of dimensions, the AF family can adopt identification criteria ranging from the union to the intersection approach.[5, 6] The AF measures are a function of the headcount of multidimensional poverty, and of the average number of deprivations suffered by the poor (and the average poverty gaps for continuous variables). The intuitiveness and easy applicability of their identification and aggregation methods are reflected in the recent decision by the UNDP to estimate members of the AF family, including the adjusted headcount ratio, M^0, for the first time for 104 countries (See Alkire and Santos, 2010). This is part of an ongoing trend of the AF measures being applied in poverty measurement as well as in other fields unrelated to poverty measurement.[7]

An immediate concern with any composite index, like those of the AF family, is that the orderings they produce, when comparing different groups, may not be robust to changes in the index's parameters.[8] For instance, in the case of the AF measures, changes in the dimensions' weights or poverty lines, as well as changes in the multidimensional cut-off, could reverse the rankings of different countries or provide contradictory results when ascertaining

[1] See for instance, Sen (2001, chapter 4), and Sen (2009, chapter 12).

[2] Ravallion (2010), among others, discusses the pros and cons of each option.

[3] For a comparative discussion of approaches to measuring multidimensional poverty, see Atkinson (2003). For a stochastic dominance approach to multidimensional poverty see Duclos et al. (2006, 2007).

[4] For instance, if considering 10 dimensions of wellbeing, a multidimensional deprivation cut-off of 5 means that a person is considered multidimensionally poor if the person is deprived in 5 or more of the 10 dimensions.

[5] According to the union approach, any person deprived in at least one dimension is considered multidimensionally poor. On the other extreme, the intersection approach demands considering as multidimensionally poor only people who are deprived in every dimension.

[6] A cut-off equal to 1 yields the union approach when all dimensions are weighted equally in the counting. Otherwise this is not necessarily true. On the other hand, a cut-off equal to the minimum weight considered always yields the union approach and vice versa.

[7] For instance, Batana (2008), Santos and Ura (2008), Alkire and Seth (2008), Battiston et al. (2009), Foster et al. (2009), Azevedo and Robles (2009), Singh (2009), Trafton (2009) and Roche (2009).

[8] For a recent articulation of this concern see, for instance, Ravallion (2010).

the direction of changes in poverty over time. With these concerns in mind, Alkire and Foster (2010) and Lasso de la Vega (2009) derived dominance conditions that, when fulfilled, ensure the robustness of comparisons to changes in the value of the multidimensional cut-off. These conditions, however, assume that weights and poverty lines remain fixed. But what if these also move? This paper derives extended conditions that, when fulfilled, ensure the robustness of comparisons to changes not just in the value of the multidimensional cut-off, but also to changes in weights and dimension-specific poverty lines. The dominance conditions are based on both the cumulative density functions and the survival functions, and combine results from both Anderson (2008) and Alkire and Foster (2010).

First, the paper provides dominance conditions for the adjusted headcount ratio, M^0, for applications with ordinal variables (e.g. as in Alkire and Santos (2010)). Then the paper also provides first-order dominance conditions for the whole Alkire-Foster family when all the variables are continuous.[9] The paper's conditions work with bivariate distributions. However, even though bivariate applications have also been popular in poverty and wellbeing analysis (e.g. Atkinson and Bourguignon (1982), Duclos et al. (2006, 2007)), recent empirical applications of the Alkire-Foster family, and other indices, consider more than two variables. Do the paper's conditions work in these circumstances? This question is answered with the following two results: 1) Traditional stochastic dominance conditions based on multivariate generalizations of Atkinson and Bourguignon (1982) are not applicable to the Alkire-Foster family for most identification criteria when three or more variables are considered. 2) Only when the poor are identified by extreme approaches, i.e. by either union or intersection, the mentioned dominance conditions apply to the Alkire-Foster family for any number of variables. In those cases, the dominance conditions depend only on the marginal distributions (union approach) or only on the joint cumulative distributions (intersection approach).

The next section briefly presents a version of the stochastic dominance conditions already derived by Alkire and Foster (2010) and Lasso de la Vega (2009) for the multidimensional cut-off. It is followed by a section introducing the new conditions for the adjusted headcount ratio. The subsequent section provides a first-order dominance condition for the whole Alkire-Foster family with continuous variables. These two previous sections work with bivariate distributions. Then the problem of applying traditional dominance conditions to three or more variables is discussed in the next section. The subsequent two sections show the peculiar cases of the union and the intersection approaches, respectively. The paper concludes with some concluding remarks.

The stochastic dominance conditions of Alkire and Foster for the adjusted headcount ratio

Notation

Consider a matrix X, whose N rows have information on the attainments of N individuals. Each column, therefore, hosts the distribution of each attainment across the population. The number of columns/variables is D. A typical attainment element of the matrix is: x_{nd} ($\in \mathbb{R}$),

[9] With the exception of the adjusted headcount ratio, the members of the Alkire-Foster family are sensitive to the gaps between the values of the variables and their specific poverty lines. Therefore these composite indices are only suitable for continuous variables.

that is, the attainment of individual n in dimension/variable d.

The identification of the multidimensionally poor has two stages. In the first stage, the poverty lines, specific to each variable, are denoted by z_d;[10] and a person is deemed poor in variable d if: $x_{nd} \leq z_d$.In the second stage, the number of deprivations is computed, weighting each deprivation with weights, w_d, such that: $w_d \in \mathbb{R}_+ \wedge \sum_d^D w_d = D$. Then the weighted number of deprivations suffered by individual n is: $c_n \equiv \sum_{d=1}^{D} w_d I(z_d > x_{nd})$.[11] If $c_n \geq k$, where $k \in \mathbb{R}_0^+$ is a multidimensional poverty cut-off such that $0 \leq k \leq D$, then individual n is said, and identified, to be multidimensionally poor.

Now the multidimensional headcount can be defined:

$$H(X; k, Z) \equiv \frac{1}{N} \sum_{n=1}^{N} I(c_n \geq k) \tag{1}$$

Also the average number of deprivations of the multidimensionally poor in the same period is defined:

$$A(X; k, Z) \equiv \frac{\sum_{n=1}^{N} I(c_n \geq k) c_n}{DNH(X; k, Z)} \tag{2}$$

Finally the adjusted headcount ratio is:

$$M^0(X; k, Z) \equiv H(X; k, Z) A(X; k, Z) = \frac{\sum_{n=1}^{N} I(c_n \geq k) c_n}{DN} \tag{3}$$

The conditions

Alkire and Foster (2010) derive a dominance condition that ensures the robustness of a comparison based on H for all values of k.[12] To derive the condition they construct a counting vector in the population by defining the variable: $a_n \equiv D - c_n$. Naturally, $a_n \in [0, D]$. Then the vector is $a := (a_1, \ldots, a_N)$. The first dominance result is the following:

$$a^i \succeq_{FD} a^j \leftrightarrow H^i \leq H^j \ \forall k \in [0, D] \tag{4}$$

where i and j denote two compared groups, and $\succeq_{FD}$means "(weakly) first-order stochastically dominates". The proof is simple: $a^i \succeq_{FD} a^j$ implies that $F^i(D-k) \leq F^i(D-k) \, \forall k \in [0, D]$. But notice that:

$F(D-k) = \frac{1}{N} \sum_{n=1}^{N} I(D - c_n \leq D - k) = \frac{1}{N} \sum_{n=1}^{N} I(c_n \geq k) = H(k)$. Hence condition (4) follows. A second result, involving the adjusted headcount ratio, is the following:

$$\left(a^i \succeq_{FD} a^j \leftrightarrow H^i \leq H^j\right) \rightarrow M^{0i} \leq M^{0j} \ \forall k \in [0, D] \tag{5}$$

[10] From a vector of poverty lines, $Z : (z_1, \ldots, z_d, \ldots, z_D)$.

[11] $I()$ is an indicator that takes the value of 1 if the expression in parenthesis is true. Otherwise it takes the value of 0.

[12] Lasso de la Vega (2009) also shows that this condition is relevant for a whole family of poverty counting measures.

The proof is straightforward for natural values of k, as Lasso de la Vega (2009) and Alkire and Foster (2010) shows. Since I consider below values for k in the real line, I prove condition (5) for continuous values of k, accordingly. The first step consists of writing $M^0(k)$ as a function of multidimensional headcounts differentials:

$$M^0(k) = \frac{1}{D}\left[H(D)D - \int_k^D tdH(t)\right] \tag{6}$$

Integrating $\int_k^D tdH(t)$ in (6), by parts, yields:

$$M^0(k) = \frac{1}{D}\left[kH(k) + \int_k^D H(t)\,dt\right] \tag{7}$$

As is clear from (7), $M^0(k)$ is a linear combination of all the headcount ratios from k to D. Hence $H^i \leq H^j \forall k \in [0, D] \rightarrow M^{0i} \leq M^{0j}\ \forall k \in [0, D]$. These conditions establish the robustness of the comparison using H and M^0 for all values of the multidimensional cut-off, k, but for a fixed set of weights and poverty lines. However, the comparison may or may not be robust to changes in either w_d, Z, or both, that may alter the distributions of a. In the next section, a dominance condition is derived for the bivariate case. Its fulfillment ensures the robustness of comparisons to all weights, poverty lines and multidimensional cut-offs.

The extended conditions for the adjusted headcount ratio for the bivariate case

An intuitive derivation

The fulfillment of the above dominance conditions requires that: $\frac{1}{N^i}\sum_{n=1}^{N^i} I(D - c_n^i \leq D - k) \leq \frac{1}{N^j}\sum_{n=1}^{N^j} I(D - c_n^j \leq D - k)\,\forall k \in [0, D]$, where the superscripts i and j denote groups as above. Notice that the cumulative functions, $F(D-k)$, bear the key characteristics that define the social welfare functions considered in the stochastic dominance literature. Both types of functions are "additively separable and symmetric with respect to individuals"(Atkinson and Bourguignon, 1982, p. 190). Moreover, they are both invariant to population replications.[13] Therefore it is possible, in principle, to integrate (or sum) by parts and derive conditions under which $F^i(D-k) \leq F^j(D-k)$ hold independently of poverty lines and weights. In this analogy the role of the individual welfare function is played by the indicator function: $I(D - c_n \leq D - k)$. Hence a first-order stochastic dominance condition for the headcount ratio could be derived by summing by parts. But since these dominance results are known,[14] in this section the condition is derived intuitively *for the case of two variables*, noticing the following two results that hold *when all variables are ordinal*:

[13] This property is implicit in the formulas of the social welfare functions considered by Atkinson and Bourguignon (1982).

[14] See, for example, Yalonetzky (2010) for the case of ordinal variables.

1. The derivative of the individual welfare function, $I\left(w_1 I\left(x_1 \leq z_1\right) + w_2 I\left(x_2 \leq z_2\right) \geq k\right)$, with respect to a unitary change in x_1, is the following difference function:

$$\begin{aligned}\frac{\Delta I\left(c_n \geq k\right)}{\Delta x_1} = & -I\left(w_2 I\left(x_2 \leq z_2\right) < k \wedge x_1 > z_1\right) I\left(x_1 + \Delta x_1 \leq z_1\right) \\ & -I\left(x_1 \leq z_1 \wedge c_n \geq k\right) I\left(x_1 + \Delta x_1 > z_1\right) I\left(c_n - w_1 I\left(x_1 \leq z_1\right) < k\right)\end{aligned} \tag{8}$$

Expression (8), which is non-positive, states that a decrease in x_1 changes the indicator function from 0 to 1 if the individual was both multidimensionally non-poor as well as non-poor in dimension 1 ($I\left(c_n^i < k \wedge x_1 > z_1\right)$) and if the decrease renders the individual poor in dimension 1 and multidimensionally poor according to the cut-off value k. It also states that an increase in x_1 changes the indicator functions from 1 to 0 if the individual was multidimensionally poor (according to k), as well as poor in dimension 1 ($I\left(c_n^i \geq k \wedge x_1 \leq z_1\right)$) and if the increase renders the individual non-poor in dimension 1 and multidimensionally non-poor according to the cut-off value k. Otherwise no change on the indicator function is produced. In either case, the difference function has a non-positive value ($\uparrow x_d \rightarrow \downarrow I$, as it were).

On the other hand, the cross-partial derivative, $\frac{\Delta^2 I(c_n \geq k)}{\Delta x_1 \Delta x_2}$, can be either positive, negative or equal to zero:

$$\frac{\Delta^2 I\left(c_n \geq k\right)}{\Delta x_1 \Delta x_2} = I\left(w_1 < k \wedge w_2 < k\right) - I\left(w_1 > k \wedge w_2 > k\right) \tag{9}$$

Expression (9) indicates the circumstances under which the difference $\frac{\Delta I(c_n \geq k)}{\Delta x_1}$ may change from 0 to -1, and vice versa, when x_2 changes. It also states the circumstances under which the difference ratio $\frac{\Delta I(c_n \geq k)}{\Delta x_2}$ may change from 0 to -1, and vice versa, when x_1 changes. The different signs that this cross-partial derivative can take reflect the different ways in which the two variables may affect each other's effect on multidimensional poverty. For instance, when poverty identification follows the intersection approach ($w_1 < k \wedge w_2 < k$), then the impact of an increase in x_1 is eliminated by a previous increase in x_2 if the person was poor to begin with (expression (22) changes value from -1 to 0). This is a case of ALEP substitution. [15] By contrast, when poverty identification follows the union approach ($w_1 > k \wedge w_2 > k$), the impact of an increase in x_1 is enhanced by a previous increase in x_2 if the person was poor to begin with (expression (22) changes value from 0 to -1). The latter is an example of ALEP complementarity, in which the cross-difference (9) is negative.

Now, in the bivariate stochastic dominance literature there are four well-established first-order conditions, all of which are relevant to the Alkire-Foster family. They all stem from the following equations:[16]

[15] For a definition of ALEP substitution and complementarity see Kannai (1980).

[16] The equivalent equations for ordinal variables are very similar. The cumulative and survival density functions need to be replaced by cumulative and survival multinomial probabilities, plus some minor adjustments. See Yalonetzky (2010).

$$\Delta W = -\int_0^{\overline{x_1}} U_1(x_1,\overline{x}_2)\,\Delta F_1(x_1)\,dx_1 - \int_0^{\overline{x_2}} U_2(\overline{x}_1,x_2)\,\Delta F_2(x_2)\,dx_2 + \int_0^{\overline{x_1}}\int_0^{\overline{x_2}} U_{12}(x_1,x_2)\,\Delta F_{12}(x_1,x_2)\,dx_1dx_2 \tag{10}$$

and:

$$\Delta W = \int_0^{\overline{x_1}} U_1(x_1,0)\,\Delta\overline{F_1}(x_1)\,dx_1 + \int_0^{\overline{x_2}} U_2(0,x_2)\,\Delta\overline{F_2}(x_2)\,dx_2 + \int_0^{\overline{x_1}}\int_0^{\overline{x_2}} U_{12}(x_1,x_2)\,\Delta\overline{F_{12}}(x_1,x_2)\,dx_1dx_2 \tag{11}$$

where U_d is the derivative of an individual welfare function U with respect to variable x_d; $\Delta W \equiv W^i - W^j$, and W is a social welfare function; $\overline{x_d}$ is the maximum value taken by variable x_d, and $\overline{F_i}, \overline{F_{ij}}$, etc. are survival functions. A condition associated with $\Delta F_{12}(x_1,x_2), \Delta F_1, \Delta F_2 \leq 0 \forall x_1, x_2 \in [0,\overline{x_1}] \times [0,\overline{x_2}]$, requires the cross-partial derivative to be negative, i.e. the two variables must be ALEP substites. An alternative condition, associated with $\Delta\overline{F_{12}}(x_1,x_2), \Delta\overline{F}_1, \Delta\overline{F}_2 \geq 0 \forall x_1, x_2 \in [0,\overline{x_1}] \times [0,\overline{x_2}]$, requires the cross-partial derivative to be positive, i.e. the variables must be ALEP complements. A third condition stems from merging the first two distributional conditions. If these are met simultaneously, then the comparison, ΔW, is robust for all individual welfare functions U, characterized by weak increasing monotonicity with respect to each variable, that is $U_1(x_1,.), U_2(x_2,.) \geq 0 \forall x_1, x_2 \in [0,\overline{x_1}] \times [0,\overline{x_2}]$. The condition is stringent, but once fulfilled it guarantees robustness for all increasingly monotonic functions irrespective of the sign of their cross-partial derivative.

The first derivative of $I(c_n \geq k)$ is non-positive. Yet a society is better-off than another one when its value for $\frac{1}{N}\sum_{n=1}^{N} I(c_n \geq k)$ is the lower. Therefore the distributional conditions related to (10) and (11) are also relevant for $\frac{1}{N}\sum_{n=1}^{N} I(c_n \geq k)$. Since the cross-partial derivatives of $I(c_n \geq k)$ can take any sign, then only the third bivariate condition applies. The dominance condition then becomes:

$$\frac{1}{N^i}\sum_{n=1}^{N^i} I\left(c_n^i \geq k\right) \leq \frac{1}{N^j}\sum_{n=1}^{N^j} I\left(c_n^j \geq k\right), \tag{12}$$

$$\forall k \in [0,2]\,\forall w_d \in \mathbb{R}_+ \wedge \sum_{d=1}^{2} w_d = 2; \forall Z \leftrightarrow$$

$$\Delta F_{12}(x_1,x_2), \Delta F_1, \Delta F_2 \leq 0 \forall x_1, x_2 \in [0,\overline{x_1}] \times [0,\overline{x_2}]$$

$$\wedge \Delta\overline{F_{12t}}(x_1,x_2), \Delta\overline{F}_1, \Delta\overline{F}_2 \geq 0 \forall x_1, x_2 \in [0,\overline{x_1}] \times [0,\overline{x_2}]$$

Combining (12) with (7) leads to the following condition:

$$
\begin{aligned}
(\frac{1}{N^i}\sum_{n=1}^{N^i} I\left(c_n^i \geq k\right) &\leq \frac{1}{N^j}\sum_{n=1}^{N^j} I\left(c_n^i \geq k\right)\forall k \in [0,2]\,; \qquad (13)\\
\forall w_d &\in \mathbb{R}_+ \wedge \sum_{d=1}^{2} w_d = 2; \forall Z \leftrightarrow\\
\Delta F_{12}\left(x_1, x_2\right), \Delta F_1, \Delta F_2 &\leq 0 \forall x_1, x_2 \in [0, \overline{x_1}] \times [0, \overline{x_2}]\\
\wedge \Delta \overline{F_{12t}}\left(x_1, x_2\right), \Delta \overline{F}_1, \Delta \overline{F}_2 &\geq 0 \forall x_1, x_2 \in [0, \overline{x_1}] \times [0, \overline{x_2}]\\
&\rightarrow M_i^0 \leq M_j^0 \forall k \in [0,2]\,; \forall w_d \in \mathbb{R}_+ \wedge \sum_{d=1}^{2} w_d = 2; \forall Z
\end{aligned}
$$

Notice that the fulfillment of the distributional conditions of (13) imply (and are implied by) dominance of society i over j on H (and on M^0) *for each and every admissible value of* k. The reason is that all the different variants of H generated by different values of k are based on individual welfare functions $(I\left(c_n \geq k\right))$ belonging to the same family of functions characterized by weak increasing monotonicity with respect to each of their arguments. Notice also that (13) is sufficiently fulfilled if the conditions on the joint cumulative and survival functions hold.

An illustration

The following is a simple example to show how these conditions work. Let there be two variables, x and y, each taking only two values, e.g. x_1 and x_2. Now consider the following joint distributions for societies A, B and C:

$$
A = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.3 & 0.1 \\ x_2 & 0.1 & 0.5 \end{array}; \quad B = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.2 & 0.2 \\ x_2 & 0.2 & 0.4 \end{array}; \quad C = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.2 & 0.15 \\ x_2 & 0.15 & 0.5 \end{array}
$$

In this simple example, with two variables each with only two values, the only values admissible for c_n, which is compared to k, are $c_n = [w_x, w_y, w_x + w_y]$. But each can be combined with a total of four sets of variable-specific poverty lines: $\left(z_x, z_y = x_1, y_1\right), \left(z_x, z_y = x_1, y_2\right),$

$\left(z_x, z_y = x_2, y_1\right), \left(z_x, z_y = x_2, y_2\right)$. The last set is not interesting because in that case clearly $H^A\left(k\right) = H^B\left(k\right) = H^C\left(k\right) = 1 \forall k \in [w_x, w_y, w_x + w_y]$. All the possible headcounts for the three societies stemming from all the relevant combinations of poverty lines, multidimensional cut-offs and weights are on Table 1.

Table 1: Multidimensional headcount ratios

Values of c_n	w_x vs w_y	z_x, z_y	H^A	H^B	H^C
w_x	$w_x > w_y$	x_1, y_1	0.4	0.4	0.35
w_x	$w_x \leq w_y$	x_1, y_1	0.5	0.6	0.5
w_x	$w_x > w_y$	x_1, y_2	0.4	0.4	0.35
w_x	$w_x \leq w_y$	x_1, y_2	1	1	1
w_x		x_2, y_1	1	1	1
w_y	$w_y > w_x$	x_1, y_1	0.4	0.4	0.35
w_y	$w_y \leq w_x$	x_1, y_1	0.5	0.6	0.5
w_y		x_1, y_2	1	1	1
w_y	$w_y > w_x$	x_2, y_1	0.4	0.4	0.35
w_y	$w_y \leq w_x$	x_2, y_1	1	1	1
$w_x + w_y$		x_1, y_1	0.3	0.2	0.2
$w_x + w_y$		x_1, y_2	0.4	0.4	0.35
$w_x + w_y$		x_2, y_1	0.4	0.4	0.35

Notice that for all possible combinations of parameters: $H^C \leq H^i, \forall i = A, B$. However the ranking between A and B depends on specific choices of parameters. For instance, with an intersection approach and poverty lines $z_x, z_y = x_1, y_1 : H^A > H^B$; whereas with a union approach and the same poverty lines: $H^A \leq H^B$.[17] Checking out the cumulative and survival functions brings out the explanation as to why C dominates A and B, why there is no dominance between A and B, and why A fares better than B with a union approach, but not with an intersection approach. The cumulative and survival functions are the following:

$$F^A = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.3 & 0.4 \\ x_2 & 0.4 & 1 \end{array} \; ; \; F^B = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.2 & 0.4 \\ x_2 & 0.4 & 1 \end{array} \; ; \; F^C = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 0.2 & 0.35 \\ x_2 & 0.35 & 1 \end{array}$$

$$\overline{F}^A = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 1 & 0.6 \\ x_2 & 0.6 & 0.5 \end{array} \; ; \; \overline{F}^B = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 1 & 0.6 \\ x_2 & 0.6 & 0.4 \end{array} \; ; \; \overline{F}^C = \begin{array}{ccc} & y_1 & y_2 \\ x_1 & 1 & 0.65 \\ x_2 & 0.65 & 0.5 \end{array}$$

Notice that the cumulative probabilities of C are never above those of A or B. Therefore in general, C cannot be poorer than A or B when the headcounts are based on identifications that involve different forms of intersections between the variables. Moreover, the survival probabilities of C are never below those of A and B. This means that for any selection of poverty lines, weights and multidimensional cut-offs, C has at least as many people as A or B who are above those thresholds, i.e. they are better-off. In other words, when the headcounts are based on identifications that involve different forms of unions between variables, C is never worse-off vis-a-vis A or B. Meanwhile, B's cumulative probabilities are never above those of A's, while A's survival probabilities are never below those of B. Hence there is no dominance relationship on the headcount (and hence on the adjusted headcount ratio) between the two societies. Their pairwise ranking depends on the identification approach chosen. A fares

[17] In this case, whether $H^A < H^B$ or $H^A = H^B$ depends on the relationship between w_x and w_y.

better than B with union approaches, because its survival probabilities are higher, whereas B fares better than A with intersection approaches because its cumulative probabilities are lower.

As the example shows, probing the stochastic dominance conditions is tantamount to testing the robustness of a pairwise ranking, based on a multidimensional headcount, to all possible alternatives of weights, variable-specific poverty lines and multidimensional cut-offs.

A first-order stochastic dominance condition for the general Alkire-Foster family using continuous variables

The general family of Alkire-Foster measures is composed of different ways of averaging normalized deprivation gaps across people and variables, each way raising the deprivation gaps to a different power stemming from the set of non-negative real numbers:

$$M^{\alpha}(X;k,Z) \equiv \frac{1}{N}\sum_{n=1}^{N} I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right) \forall\alpha \in \mathbb{R}_0^+, \tag{14}$$

where, for a value y, $[y]_+ = yI(y \geq 0)$. Notice that when $\alpha = 0 : M^0 = \frac{\sum_{n=1}^{N} I(c_n \geq k)c_n}{DN}$, as in (3).

M^{α} can now be regarded as a social welfare function, like W, and the respective individual welfare function, U, is: $I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)$. In the bivariate case, for the derivation of a first-order dominance condition, it suffices to look at U_d and the cross-partial derivative, as in the previous section. Moreover, if it is found that U_{12} can take any sign, then the first-order dominance condition for the Alkire-Foster family of measures depends also on both the cumulative density and survival functions.

The partial derivative of $I(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)$ with respect to x_t is:

$$\begin{aligned}\frac{dI(c_n \geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_t} &= \left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)\frac{dI(c_n \geq k)}{dx_t} \\ &+ I(c_n \geq k)\frac{d\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_t},\end{aligned} \tag{15}$$

where:

$$\begin{aligned}\frac{dI(c_n \geq k)}{dx_t} &= -I(c_n < k \wedge x_t > z_t)\, I(c_n + w_t I(x_t - dx_t \leq z_t) \geq k)\infty \\ &\quad -I\left(c_n^i \geq k \wedge x_t \leq z_t\right) I\left(c_n^i - w_t I(x_t + dx_t > z_t) < k\right)\infty.\end{aligned} \tag{16}$$

That is, $\frac{dI(c_n \geq k)}{dx_t} = 0$, if an infinitesimal change, dx_t, does not change the individual's multidimensional poverty (or non-poverty) status. Otherwise $\frac{dI(c_n \geq k)}{dx_t} = -\infty$. Also:

$$\frac{d\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_t}=-\alpha\frac{w_t}{Dz_t}\left[\frac{z_t-x_{nt}}{z_t}\right]^{\alpha-1}I\left(z_t>x_{nt}\right). \tag{17}$$

That is, an infinitesimal increase in x_t decreases the sum of censored gaps $\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}$ if $z_t>x_{nt}$. Otherwise: $\frac{d\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_t}=0$.

Considering (16) and (17) it is clear that: $\frac{dI(c_n\geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_t}\leq 0$ and that a society is better off the lower M^{α} is. Therefore, as in the previous section, the distributional conditions associated with (10) and (11) are also relevant for M^{α}, in the bivariate case. The simplest cross-partial derivative is:

$$\begin{aligned}\frac{d^2I\left(c_n\geq k\right)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_tdx_s} &= \left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)\frac{d^2I\left(c_n\geq k\right)}{dx_tdx_s} \\ &\quad -\frac{dI\left(c_n\geq k\right)}{dx_t}\alpha\frac{w_s}{Dz_s}\left[\frac{z_s-x_{ns}}{z_s}\right]^{\alpha-1}I\left(z_s>x_{ns}\right) \\ &\quad -\frac{dI\left(c_n\geq k\right)}{dx_s}\alpha\frac{w_t}{Dz_t}\left[\frac{z_t-x_{nt}}{z_t}\right]^{\alpha-1}I\left(z_t>x_{nt}\right) \\ &\quad +I\left(c_n\geq k\right)\frac{d^2\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_tdx_s}\end{aligned} \tag{18}$$

From (16) it is clear that $\frac{dI(c_n\geq k)}{dx_s}\leq 0$. It is also clear from (17) that $\frac{d^2\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_tdx_s}=0$. Hence the fourth element of the right-hand side of (18) is equal to zero, and both the third and the second element are non-negative. The first element, however, can take any sign because it is the product of a non-negative sub-element, $\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)$, and $\frac{d^2I(c_n\geq k)}{dx_tdx_s}$. In the bivariate case, $\frac{d^2I(c_n\geq k)}{dx_tdx_s}$ is very similar to the ordinal-variables formulation in (9), with the difference that, on the right-hand side, Δx needs to be replaced with dx and the whole right-hand side must be multiplied by ∞. Therefore: $\frac{d^2I(c_n\geq k)}{dx_tdx_s}=-\infty\vee 0\vee\infty$. Consequently also $\frac{d^2I(c_n\geq k)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)}{dx_tdx_s}=-\infty\vee 0\vee\infty$.

Since the cross-partial derivative can take any sign, the reasoning of the previous section leads to the conclusion that the first-order stochastic dominance condition for M^{α} $\forall\alpha\in\mathbb{N}$, is very similar to the one for the multidimensional headcount and ordinal variables, i.e. (12):

$$\begin{aligned} M_i^{\alpha}(X;k,Z) &\leq M_j^{\alpha}(X;k,Z)\,\forall k\in[0,2], \qquad (19)\\ \forall\alpha &\in \mathbb{R}_0^{+}\forall w_d\in\mathbb{R}_+\wedge\sum_{d=1}^{2}w_d=2;\forall Z\leftrightarrow\\ \Delta F_{12}(x_1,x_2),\Delta F_1,\Delta F_2 &\leq 0\forall x_1,x_2\in[0,\overline{x_1}]\times[0,\overline{x_2}]\\ \wedge\Delta\overline{F_{12}}(x_1,x_2),\Delta\overline{F}_1,\Delta\overline{F}_2 &\geq 0\forall x_1,x_2\in[0,\overline{x_2}]\times[0,\overline{x_2}] \end{aligned}$$

According to condition (19), multidimensional poverty in society i is never higher than in j, as measured by any member of the Alkire-Foster family in the bivariate case, if and only if the joint cumulative density function of i is never above that of j, and the joint survival function of i is never below that of j, for all choices of specific poverty lines, weights, multidimensional poverty cut-offs and values for the α parameters.

The problem with three or more variables

The above results cannot be extended to the case of three or more variables, unless indentification of the poor is undertaken either using the intersection or the union approach. In this section, I show why the results cannot be extended to several variables and intermediate identification approaches. In the next two sections I show why and how the approach is applicable to any multivariate distribution as long as the extreme identification approaches are considered.

The reason why the results are not applicable to cases of three, or more, variables (save the two mentioned expections) is that with more than two variables, the multivariate- version of (10) and (11), for first-order dominance conditions, require checking the signs of all cross-partial derivatives involving all combinations of variables (i.e. including three and more variables). Existing multivariate conditions that work on cumulative and survival functions can handle any sign of cross-partial derivatives involving even numbers of variables. However, for odd numbers of variables (e.g. 1, 3, 5, etc.) the conditions only apply to non-negative cross-partial derivatives (or non-positive, in the case of poverty functions). This is clear by examining the multivariate versions of (10) and (11):[18]

[18] The equivalent equations for ordinal variables are very similar. The cumulative and survival density functions need to be replaced by cumulative and survival multinomial probabilities, plus some minor adjustments. See Yalonetzky (2010).

$$\begin{aligned}\Delta W &= -\sum_{d=1}^{D}\int_0^{\overline{x_d}} U_d\left(x_d,\ldots,\overline{x}_{s\neq d}\right)\Delta F_d dx_d \qquad (20)\\ &+\sum_{d=1}^{D-1}\sum_{s=d+1}^{D}\int_0^{\overline{x_s}}\int_0^{\overline{x_d}} U_{ds}\left(x_d,x_s,\ldots,\overline{x}_{t\neq s,d}\right)\Delta F_{ds}\left(x_d,x_s\right)dx_d dx_s\\ &-\sum_{d=1}^{D-2}\sum_{s=d+1}^{D-1}\sum_{t=s+1}^{D}\int_0^{\overline{x_t}}\int_0^{\overline{x_s}}\int_0^{\overline{x_d}} U_{dst}\left(x_d,x_s,x_t,\ldots,\overline{x}_{u\neq d,s,t}\right)\Delta F_{dst}\left(x_d,x_s,x_t\right)dx_d dx_s dx_t\cdots\\ &+(-1)^D\int_0^{\overline{h_D}}\cdots\int_0^{\overline{h_1}} U_{1\ldots D}\Delta F\left(x_1,\ldots,x_D\right)dx_1\ldots dx_D\end{aligned}$$

and:

$$\begin{aligned}\Delta W &= \sum_{d=1}^{D}\int_0^{\overline{x_d}} U_d\left(x_d,0,\ldots,0\right)\Delta\overline{F_d}dx_d \qquad (21)\\ &+\sum_{d=1}^{D-1}\sum_{s=d+1}^{D}\int_0^{\overline{x_s}}\int_0^{\overline{x_d}} U_{ds}\left(x_d,x_s,0,\ldots,0\right)\Delta\overline{F_{ds}}\left(x_d,x_s\right)dx_d dx_s\\ &+\sum_{d=1}^{D-2}\sum_{s=d+1}^{D-1}\sum_{t=s+1}^{D}\int_0^{\overline{x_t}}\int_0^{\overline{a_s}}\int_0^{\overline{a_d}} U_{dst}\left(x_d,x_s,x_t,0,\ldots,0\right)\Delta\overline{F_{dst}}\left(x_d,x_s,x_t\right)dx_d dx_s dx_t\cdots\\ &+\int_0^{\overline{h_D}}\cdots\int_0^{\overline{h_1}} U_{1\ldots D}\Delta\overline{F}_{12\ldots D}\left(x_1,\ldots,x_D\right)dx_1\ldots dx_D\end{aligned}$$

where the notation is the same as in (10) and (11).[19] A condition associated with $\Delta F\left(x_1,\ldots,x_D\right),\ldots,\Delta F_{dst}\left(x_d,x_s,x_t\right),\ldots\Delta F_d\leq 0\forall x_1,\ldots,x_D\in[0,\overline{x_1}]\times\ldots[0,\overline{x_D}]$, requires that the cross-partial derivatives alternate signs starting with $U_d\left(x_d,\ldots,\overline{x}_{s\neq d}\right)\geq 0$, followed by $U_{ds}\left(x_d,x_s,\ldots,\overline{x}_{t\neq d,s}\right)\leq 0$, and so on until $U_{1\ldots D}\leq 0$ if D is an even number, or $U_{1\ldots D}\geq 0$ otherwise. An alternative condition, associated with $\Delta\overline{F}\left(x_1,\ldots,x_D\right),\ldots,\Delta\overline{F_{dst}}\left(x_d,x_s,x_t\right),\ldots\Delta\overline{F}_d\geq 0\forall x_1,\ldots,x_D\in[0,\overline{x_1}]\times\ldots[0,\overline{x_D}]$, requires that all cross-partial derivatives be non-negative. A third condition stems from the merger of the first two distributional conditions. If the two first conditions are met, then the comparison, ΔW, is robust for all individual welfare functions U, characterized by weak increasing monotonicity with respect to each variable, $\left(U_d\left(x_d,\ldots,\overline{x}_{s\neq d}\right)\geq 0\forall d\right)$ and by positively (or zero) signed cross-partial derivatives for odd numbers of variables. Hence, for the conditions stemming from (20) and (21) to be suitable for the derivation of multivariate versions of (12) and (19), it is necessary that the odd cross-partial derivatives of both $I\left(c_n\geq k\right)$ (in the ordinal case) and $I\left(c_n\geq k\right)\left(\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}\right)$ (in the continuous case) be non-positive (because it is a

[19] The result (20) has been shown by Crawford (2005), although it was alluded to Hadar and Russell (1974). The result (21) is a simple multidimensional extension of the three dimensional derivation by Anderson (2008).

poverty function). By examining the simplest cross-partial derivatives it is easy to realize that the conditions are not suitable. First, the multivariate equivalent of (8), i.e. the partial derivative is non-positive:

$$\begin{aligned} \frac{\Delta I\left(c_n \geq k\right)}{\Delta x_d} &= \\ & -I\left(c_n < k \wedge x_d > z_d\right) I\left(c_n + w_d I\left(x_d - \Delta x_d \leq z_d\right) \geq k\right) \\ & -I\left(c_n \geq k \wedge x_d \leq z_d\right) I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) < k\right) \end{aligned} \tag{22}$$

The derivative (22) does not pose any problem. The multivariate equivalent of (9) is more complicated (because it depends not just on the two differentiation variables). It can also take any sign:

$$\begin{aligned} & \frac{\Delta I^2\left(c_n \geq k\right)}{\Delta x_d \Delta x_s} \\ = \;& I\left(c_n < k \wedge x_d > z_d \wedge x_s > z_s\right) I\left(c_n + w_d I\left(x_d + \Delta x_d \leq z_d\right) < k \wedge c_n + w_s I\left(x_s + \Delta x_s \leq z_s\right) < k\right) \\ & I\left(c_n + w_d I\left(x_d + \Delta x_d \leq z_d\right) + w_s I\left(x_s + \Delta x_s \leq z_s\right) \geq k\right) \\ & +I\left(c_n < k \wedge x_d > z_d \wedge x_s < z_s\right) I\left(c_n + w_d I\left(x_d + \Delta x_d \leq z_d\right) \geq k\right) \\ & I\left(c_n + w_d I\left(x_d + \Delta x_d \leq z_d\right) - w_s I\left(x_s + \Delta x_s > z_s\right) < k\right) \\ & +I\left(c_n < k \wedge x_d < z_d \wedge x_s > z_s\right) I\left(c_n + w_s I\left(x_s + \Delta x_s \leq z_s\right) \geq k\right) \\ & I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) + w_s I\left(x_s + \Delta x_s \leq z_s\right) < k\right) \\ & -I\left(c_n \geq k \wedge x_d < z_d \wedge x_s < z_s\right) I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) \geq k \wedge c_n - w_s I\left(x_s + \Delta x_s > z_s\right) \geq k\right) \\ & I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) - w_s I\left(x_s + \Delta x_s > z_s\right) < k\right) \\ & -I\left(c_n \geq k \wedge x_d < z_d \wedge x_s > z_s\right) I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) < k\right) \\ & I\left(c_n - w_d I\left(x_d + \Delta x_d > z_d\right) + w_s I\left(x_s + \Delta x_s \leq z_s\right) \geq k\right) \\ & -I\left(c_n \geq k \wedge x_d > z_d \wedge x_s < z_s\right) I\left(c_n - w_s I\left(x_s + \Delta x_s > z_s\right) < k\right) \\ & I\left(c_n + w_d I\left(x_d + \Delta x_d \leq z_d\right) - w_s I\left(x_s + \Delta x_s > z_s\right) \geq k\right) \end{aligned} \tag{23}$$

With some further manipulation one can show that $\frac{\Delta I^3(c_n \geq k)}{\Delta x_d \Delta x_s \Delta x_t}$ can also take any sign (not just non-positive). Hence neither (20) nor (21) provide suitable dominance conditions for the class of poverty functions based on $I\left(c_n \geq k\right)$. Likewise, in the case of $I\left(c_n \geq k\right)\left(\sum_{d=1}^{D} \frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_+^{\alpha}\right)$ the cross-partial derivative involving three variables is:

$$\frac{d^3 I\left(c_n \geq k\right)\left(\sum_{d=1}^{D} \frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_{+}^{\alpha}\right)}{dx_t dx_s dx_r} = \left(\sum_{d=1}^{D} \frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_{+}^{\alpha}\right)\frac{d^3 I\left(c_n \geq k\right)}{dx_t dx_s dx_r} \tag{24}$$

$$-\frac{d^2 I\left(c_n \geq k\right)}{dx_t dx_s}\alpha\frac{w_r}{D z_r}\left[\frac{z_r - x_{nr}}{z_r}\right]^{\alpha-1} I\left(z_r > x_{nr}\right)$$

$$-\frac{dI^2\left(c_n \geq k\right)}{dx_t dx_r}\alpha\frac{w_s}{D z_s}\left[\frac{z_s - x_{ns}}{z_s}\right]^{\alpha-1} I\left(z_s > x_{ns}\right)$$

$$-\frac{dI^2\left(c_n \geq k\right)}{dx_s dx_r}\alpha\frac{w_t}{D z_t}\left[\frac{z_t - x_{nt}}{z_t}\right]^{\alpha-1} I\left(z_t > x_{nt}\right)$$

Since any of the three pairwise cross-partial derivatives in (24), as well as $\frac{d^3 I(c_n \geq k)}{dx_t dx_s dx_r}$, can take any sign, then already the odd cross-partial derivative with three variables can also take any sign. Hence the aforementioned dominance conditions are not applicable to the Alkire-Foster measures when more than two variables are considered together with indentification approaches different from either intersection or union. The next sections explain why similar conditions are suitable for the two extreme forms of identification.

The peculiar case of the union approach

The following is an interesting result among members of the Alkire-Foster family:

$M^{\alpha}(X; \min\{w_d\}, Z) = M^{\alpha}(X; 0, Z)\, \forall \alpha \in \mathbb{R}_0^{+}$. That is, the measurement of poverty using the Alkire-Foster family under the union approach (i.e. considering as multidimensionally poor anybody who is poor in at least one variable) is equal to the weighted average of all specific poverty gaps across the population:

$$M^{\alpha}(X; 0, Z) = \frac{1}{N}\sum_{n=1}^{N}\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_{+}^{\alpha} \forall \alpha \in \mathbb{R}_0^{+}. \tag{25}$$

Now notice that:

$$\frac{d\sum_{d=1}^{D} \frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_{+}^{\alpha}}{dx_t} = -\alpha\frac{w_t}{D z_t}\left[\frac{z_t - x_{nt}}{z_t}\right]_{+}^{\alpha-1} \tag{26}$$

$$\frac{d^2\sum_{d=1}^{D} \frac{w_d}{D}\left[\frac{z_d - x_{nd}}{z_d}\right]_{+}^{\alpha}}{dx_t dx_s} = 0$$

That is, the cross-partial derivatives are all equal to zero. In this situation a first-order dominance condition can be derived without recourse to either the joint cumulative density or the joint survival functions. Only the marginal distributions matter. In other words, the way the variables are associated in the compared populations is irrelevant for the dominance conditions.[20] Considering (20) and (21), a first-order dominance condition for $M^{\alpha}(X;0,Z)$ is:

$$\begin{aligned} M_i^{\alpha}(X;0,Z) &\leq M_j^{\alpha}(X;0,Z), \forall \alpha \in \mathbb{R}_0^+ \quad (27) \\ \forall w_d &\in \mathbb{R}_+ \wedge \sum_{d=1}^{D} w_d = D; \forall Z \leftrightarrow \Delta F_d \leq 0 \forall x_1, \ldots, x_D \in [0, \overline{x_1}] \times \ldots [0, \overline{x_D}] \end{aligned}$$

Finally, since $M^{\alpha}(X;\min\{w_d\},Z) = M^{\alpha}(X;0,Z) \forall \alpha \in \mathbb{N}$, then condition (27) applies to $M^{\alpha}(X;\min\{w_d\},Z)$. Thus, conditions from (20) and (21) are applicable to the union approach. One just needs to set all the cross-partial derivatives equal to zero and notice that it does not make a difference to use the univariate cumulative distributions or the univariate survival functions. The first-order dominance condition requires testing for first-order dominance over each variable separately and declaring dominance only when one society dominates another one in each and every variable, independently.

The multidimensional headcount

In the case of the multidimensional headcount, a similar condition ensues, because notice that the cross-partial derivatives of $I(c_n \geq k)$ are also zero for the union approach. In this approach, $I(c_n \geq k) = I(c_n \geq \min\{w_d\}) = I(\exists s \mid x_{ns} < z_s)$. Hence, in the ordinal case:

$$\frac{\Delta I(\exists s \mid x_{ns} < z_s)}{\Delta x_s} = -I(x_{ns} > z_s) I(x_{ns} + \Delta x_s < z_s) - I(x_{ns} < z_s) I(x_{ns} + \Delta x_s > z_s) \quad (28)$$

Then, clearly, $\frac{\Delta^2 I(\exists s | x_{ns} < z_s)}{\Delta x_s \Delta x_t} = 0$, and so all the other cross-partial derivatives are equal to zero.[21] Hence, using (20) (or (21)) the following condition is derived for the multidimensional headcount, using the union approach:

$$\begin{aligned} \frac{1}{N^i} \sum_{n=1}^{N^i} I\left(c_n^i \geq \min\{w_d\}\right) &\leq \frac{1}{N^j} \sum_{n=1}^{N^j} I\left(c_n^j \geq \min\{w_d\}\right), \quad (29) \\ \forall w_d &\in \mathbb{R}_+ \wedge \sum_{d=1}^{D} w_d = D; \forall Z \leftrightarrow \Delta F_d(x_d) \leq 0 \forall x_d \in [0, \overline{x_d}] \end{aligned}$$

The peculiar case of the intersection approach

In the case of the intersection approach: $I(c_n = D) = I(\forall s : x_{ns} < z_s)$.Then, in the ordinal case:

[20] This is, by the way, the key feature of the fourth first-order dominance condition that can be derived for D variables; namely, ALEP neutrality.

[21] In the continuous case the differences, Δ, in (28) are replaced by differential numbers, e.g. dx_{ns}, and the right-hand side is multiplied by ∞.

$$\frac{\Delta I\left(\forall s: x_{ns}<z_s\right)}{\Delta x_s} = -I\left(\forall t\neq s: x_{nt}<z_t\right) \left[I\left(x_{ns}>z_s\right)I\left(x_{ns}+\Delta x_s<z_s\right)+I\left(x_{ns}<z_s\right)I\left(x_{ns}+\Delta x_s>z_s\right)\right] \tag{30}$$

$$\frac{\Delta I^2\left(\forall s: x_{ns}<z_s\right)}{\Delta x_s \Delta x_r} = I\left(\forall t\neq s,r: x_{nt}<z_t\right) \left[I\left(x_{nr}>z_r\right)I\left(x_{nr}+\Delta x_r<z_r\right)+I\left(x_{nr}<z_r\right)I\left(x_{nr}+\Delta x_r>z_r\right)\right] \left[I\left(x_{ns}>z_s\right)I\left(x_{ns}+\Delta x_s<z_s\right)+I\left(x_{ns}<z_s\right)I\left(x_{ns}+\Delta x_s>z_s\right)\right] \tag{31}$$

By differentiating $\frac{\Delta I^2(\forall s: x_{ns}<z_s)}{\Delta x_s \Delta x_r}$ further with respect to the remaining variables it is easy to spot the pattern whereby, in the intersection approach, the cross-partial differences alternate signs. The odd derivatives have non-positive signs and the even derivatives have non-negative signs. Hence the poverty-function equivalent of condition (20) applies to the multidimensional headcount under the intersection approach.The following condition is derived:

$$\frac{1}{N^i}\sum_{n=1}^{N^i} I\left(c_n^i = D\right) \leq \frac{1}{N^j}\sum_{n=1}^{N^j} I\left(c_n^j = D\right), \forall w_d \in \mathbb{R}_+ \wedge \sum_{d=1}^{D} w_d = D; \forall Z \leftrightarrow \Delta F_{1,\ldots,D}\left(x_1,\ldots,x_D\right),\ldots,\Delta F_d\left(x_d\right) \leq 0 \forall x_1,\ldots,x_D \in [0,\overline{x_1}]\times\ldots 0,\overline{x_D}] \tag{32}$$

Notice that for (32) to be fulfilled it suffices to show that $\Delta F_{1,\ldots,D}\left(x_1,\ldots,x_D\right) \leq 0$ $\forall x_1,\ldots,x_D \in [0,\overline{x_1}]\times\ldots 0,\overline{x_D}]$ holds. Combining (32) with (7) yields the following condition for the adjusted headcount ratio under the intersection approach:

$$\Delta F_{1,\ldots,D}\left(x_1,\ldots,x_D\right),\ldots,\Delta F_d\left(x_d\right) \leq 0 \forall x_1,\ldots,x_D \in [0,\overline{x_1}]\times\ldots 0,\overline{x_D}] \rightarrow M_i^0\left(X;D,Z\right) \leq M_j^0\left(X;D,Z\right); \forall w_d \in \mathbb{R}_+ \wedge \sum_{d=1}^{D} w_d = D; \forall Z$$

The AF measures for continuous variables

In the case of AF indices for continuous variables:

$$M^{\alpha}\left(X;D,Z\right) = \frac{1}{N}\sum_{n=1}^{N} I\left(\forall s: x_{ns}<z_s\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha} \forall \alpha \in \mathbb{R}_0^+. \tag{33}$$

The partial and the pairwise cross-partial derivatives of $I\left(\forall s: x_{ns}<z_s\right)\sum_{d=1}^{D}\frac{w_d}{D}\left[\frac{z_d-x_{nd}}{z_d}\right]_+^{\alpha}$ are in (15) and (18), respectively. When $c_n^i = D$, (15) is non-positive and (18) is non-negative,

because $\frac{dI(c_n=D)}{dx_d}$ is non-positive and $\frac{d^2I(c_n\geq k)}{dx_t dx_s}$ is non-negative. The latter is non-negative because it has the same sign as (31).[22] Then cross-partial derivatives involving more variables alternate sign. Odd cross-partial derivatives, like $\frac{d^3I(c_n\geq k)}{dx_t dx_s dx_r}$, are non-positive and even derivatives are non-negative. Hence the poverty-function equivalent of condition (20) applies to the whole Alkire-Foster family for continuous variables under the intersection approach. The following condition ensues:

$$\begin{aligned} M_i^\alpha (X; D, Z) &\leq M_j^\alpha (X; D, Z), \forall \alpha \in \mathbb{R}_0^+ \forall w_d \in \mathbb{R}_+ \wedge \sum_{d=1}^{D} w_d = D; \forall Z \\ &\leftrightarrow \Delta F_{1,\ldots,D}(x_1, \ldots, x_D), \ldots, \Delta F_d(x_d) \leq 0 \forall x_1, \ldots, x_D \in [0, \overline{x_1}] \times \ldots 0, \overline{x_D}] \end{aligned} \quad (34)$$

Notice that for (34) to be fulfilled it suffices to show that $\Delta F_{1,\ldots,D}(x_1, \ldots, x_D) \leq 0$ $\forall x_1, \ldots, x_D \in [0, \overline{x_1}] \times \ldots 0, \overline{x_D}]$ holds.

Concluding remarks

This paper builds on the work of Alkire and Foster (2010) and Lasso de la Vega (2009) in order to derive, first, a first-order stochastic dominance conditions for poverty comparisons using the multidimensional headcount ratio, in applications with two ordinal variables. When fulfilled, these conditions ensure that the poverty comparison is robust to any poverty line, any weighting of the variables and any choice of the multidimensional cut-off. Secondly, the paper shows that similar conditions exist for the whole family of AF mesures, i.e. including those measures that work with continuous variables. The conditions are stringent, but they provide the maximum degree of robustness in poverty comparisons using these measures.

When three or more variables are considered, the traditional dominance conditions used in this paper are not appropriate for poverty counting measures of the AF family, except when extreme poverty identification approaches are considered. The reason is that the conditions do not contemplate the possibility that certain cross-partial derivatives may take different signs, specifically, those that stem from differentiating the individual poverty function with respect to an *odd* number of variables. For intermediate identification approaches, the odd cross-partial derivatives of the AF family can take any sign. Hence traditional dominance conditions are not applicable.

However, for the extreme identification approaches, this paper shows that there are suitable dominance conditions for any number of variables. In the case of the union approach, a country whose marginal distributions first-order dominate will not exhibit higher poverty measured by any index of the AF family, including the multidimensional headcount ratio. For the union approach, the joint distribution of variables is not necessary when testing for first-order dominance. In the case of the intersection approach, a country whose cumulative joint and marginal distributions first-order dominate will not exhibit higher poverty according to any of the AF measures, including the headcount. In this particular case, dominance

[22] The difference are between $\frac{\Delta I^2(\forall s: x_{ns}<z_s)}{\Delta x_s \Delta x_r}$ and $\frac{d^2I(\forall s: x_{ns}<z_s)}{dx_t dx_s}$ the differential numbers and the ∞ in the latter's formula.

over the joint distribution of *all* the variables involved suffices to ensure the full robustness of the poverty comparison to changes in any of the parameters of the AF measures, including the choice of family member (the value of α).

What's left for poverty comparisons using AF measures, intermediate identification approaches and several variables? In these circumstances, traditional dominance conditions cannot ascertain robustness when fulfilled. However, by looking at (20) or (21), it is clear that when they are not fulfilled (e.g. the joint cumulative or survival distributions cross), ordinal poverty comparisons with the AF family, are not robust. Since one would ideally also want conditions that ensure robustness, the latter assessment is not very satisfactory. Hence pending research should explore alternative robustness criteria for ordinal poverty comparisons with counting measures like the AF family, beyond the traditional tools developed since the seminal contribution of Atkinson and Bourguignon (1982).

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