# Poverty comparisons with cardinal and ordinal attributes

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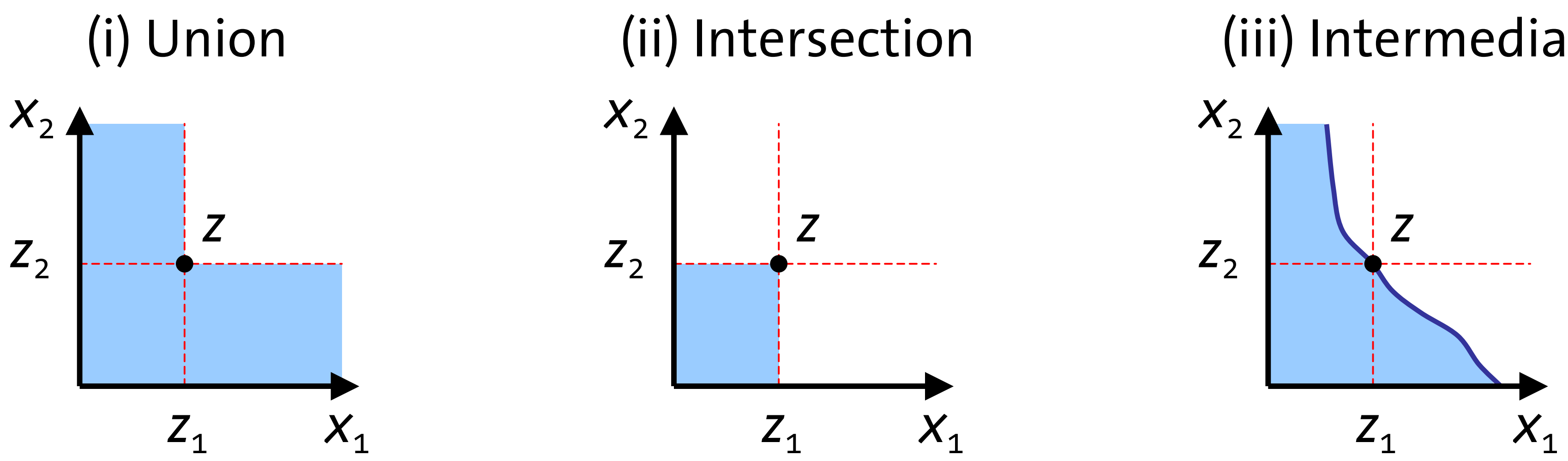
## Problem 1: How identify the poor?

- Given a *poverty bundle* $z = (z_1, z_2, \ldots)$, a threshold for each attribute, how do we determine who is poor and who is not?




- Ideally, leave the choice to the practitioner (see also Duclos, Sahn & Younger, 2006 ; Alkire & Foster, 2008)

## Problem 2: How give priority to the worse off poor?

- Consider a multidimensional Foster-Greer-Thorbecke index:

$$\Sigma_i \Pi_j (z_j - x^i_j)^{\alpha_j} \qquad \text{(the sum over the poor only)}$$

where $x^i_j$ is individual *i*'s amount of attribute *j*

- Example with poverty bundle $z = (10, 10)$ and weights $\alpha_1 = \alpha_2 = 1$
  - Individual 1 has bundle (4, 6) ; individual 2 has bundle (6.5, 4)
  - According to the index individual 1 is worse off than 2

- Assume an extra 0.5 of the first attribute can be given away
  - Worse off individual 1 gets it → poverty decreases by 2
  - Better off individual 2 gets it → poverty decreases by 3
  - Undesirable conclusion: give priority to the better off!

## Problem 3: How deal with ordinal data?

- Ordinal: e.g., health, housing, education,...
  - Example: "Would you say your health in general is 0 = poor, 1 = fair, 2 = good, 3 = very good, 4 = excellent?"
- Problem in defining priority
  - Cardinal: "Should $30 go to someone with $100 or to someone with $200?" → meaningful question
  - Ordinal: "Should 1 point of health go to someone with health 2 or to someone with health 3?" → **not** a meaningful question
- See also Alkire & Foster (2008) and Bossert, Chakravarty & D'Ambrosio (2009)

# Notation

- Set of attributes $C \cup O$ (remember *problem 3*)
  - Each individual has an attribute bundle $x = (x_C, x_O)$ in $B$
    - $x_C$ the vector of *cardinal* attributes (positive real numbers)
    - $x_O$ the vector of *ordinal* attributes (integers, 0, 1, 2, 3,...)
- Fixed poverty bundle $z$ in $B$
- A distribution $X = (x^1, x^2,...)$ is an element of $D$
- Poverty ranking $\succcurlyeq$ ("better than" relation) on $D$

## Axiom 1: Additive representability (AR)

**AR:** There exists a continuous function $\pi : B \longrightarrow \mathbb{R}$ such that, for all $X$ and $Y$ in $D$,

$$X \succcurlyeq Y \quad \text{if and only if} \quad \frac{1}{n_X} \sum_{i=1}^{n_X} \pi(x^i) \leq \frac{1}{n_Y} \sum_{i=1}^{n_Y} \pi(y^i)$$

- **AR** is a strong axiom: $\succcurlyeq$ has to be (i) continuous, (ii) anonymous, (iii) separable, (iv) replication invariant

- **AR** is strong, but quite common in the literature (see, e.g., Foster & Shorrocks, 1991 ; Tsui, 2002 ; Bourguignon & Chakravarty, 2003)

## Axiom 2: Focus (F)

- Remember *problem 1* (how to identify the poor)

- We define the set of poor in $X$ as $P = \{ i \mid x^i \prec z \}$
  - Given **AR**, the poor are those with $\pi(x) > \pi(z)$

**F:** for all $X$ in $D$, if $Y$ is obtained from $X$ by a change in the bundle of a non-poor while keeping her non-poor, then $X \sim Y$

- Given **F**, the function $\pi$ is constant for all $x$ such that $\pi(x) \leq \pi(z)$

## Axiom 3: Monotonicity (M)

**M:** for all $X$ and $Y$ in $D$ and for all poor $i$, we have that if $x^i > y^i$ and $x^j = y^j$ for all $j \neq i$, then $X \succ Y$

- Given **AR** and **M**, the function $\pi$ is strictly decreasing in each attribute for bundles $x$ such that $\pi(x) > \pi(z)$

## Axiom 4: Priority (P = CP & OP)

- Remember *problem 2* (how to give priority to the worse off poor)

**CP:** for all $X$ in $D$,

– for all $\delta = (\delta_C, \delta_O)$ in $B$ with $\delta_C > 0$ and $\delta_O = 0$

– for all poor $i$ and $j$ with $x^i \succ x^j$

we have $( \dots, x^i, \dots, x^j + \delta, \dots ) \succ ( \dots, x^i + \delta, \dots, x^j, \dots )$

**OP:** for all $X$ in $D$,

– for all $\delta = (\delta_C, \delta_O)$ with $\delta_C = 0$ and $\delta_O > 0$

– for all poor $i$ and $j$ with $x^i \succ x^j$ and $x^i_k = x^j_k$ for all $k$ for which $\delta_{O,k} > 0$

we have $( \dots, x^i, \dots, x^j + \delta, \dots ) \succ ( \dots, x^i + \delta, \dots, x^j, \dots )$

# Axiom 4: Priority (P = CP & OP)

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## Result

A poverty ranking $\succcurlyeq$ satisfies **AR**, **F**, **M** and **P** iff there exist

- weights $w_k > 0$ for each cardinal attribute $k$ in $C$,
- strictly increasing functions $v_k : \mathbb{N} \to \mathbb{R}$, with $v_k(0) = 0$, for each ordinal attribute $k$ in $O$,
- a continuous function $f : \mathbb{R}_+ \to \mathbb{R}$,
  - with $f(r) = f(\zeta)$ for each $r \geq \zeta = \Sigma_{k \in C} w_k z_k + \Sigma_{k \in O} v_k(z_k)$,
  - that is strictly convex and strictly decreasing on $[0, \zeta]$,

such that, for all $X$ and $Y$ in $D$, we have $X \succcurlyeq Y$ iff

$$\frac{1}{n_X} \sum_{i=1}^{n_X} f\Big( \sum_{k \in C} w_k x^i_k + \sum_{k \in O} v_k(x^i_k) \Big) \leq \frac{1}{n_Y} \sum_{i=1}^{n_Y} f\Big( \sum_{k \in C} w_k y^i_k + \sum_{k \in O} v_k(y^i_k) \Big)$$

## Special case: All attributes cardinal

$$X \succcurlyeq Y \qquad \text{iff} \qquad \frac{1}{n_X}\sum_{i=1}^{n_X} f\Big(\sum_{k\in C} w_k x^i_k\Big) \;\le\; \frac{1}{n_Y}\sum_{i=1}^{n_Y} f\Big(\sum_{k\in C} w_k y^i_k\Big)$$

- Satisfies:
  - Weak uniform majorization principle: if $X \neq Y$ and $X = YQ$ with $Q$ a non-permutation bistochastic matrix, then $X \succcurlyeq Y$
  - Correlation increasing majorization principle
    *Example with z = (10, 10):* $\big((\mathbf{5}, 6), (\mathbf{9}, 2)\big) \succ \big((\mathbf{9}, 6), (\mathbf{5}, 2)\big)$

## A weaker form of priority (work in progress)

- Cardinal priority (**CP**) is a demanding normative principle
  - It asks us to disregard "efficiency costs" caused by diminishing returns to well-being

- Example:
  - Suppose $x^i \succ x^j$, but individual $i$ has only 10 units of attribute 1, while individual $j$ has 100 units
  - According to **CP**, if an extra unit of attribute 1 can be given away, then it should go to individual $j$

## A weaker form of priority (work in progress)

- Consider a weaker version of **CP**:

For all $X$ in $D$,

– for all $\delta = (\delta_C, \delta_O)$ in $B$ with $\delta_C > 0$ and $\delta_O = 0$

– for all poor $i$ and $j$ with $x^i \succ x^j$ and $x^i_k \geq x^j_k$ for all $k$ for which $\delta_{C,k} > 0$

we have $(\ldots, x^i, \ldots, x^j + \delta, \ldots) \succ (\ldots, x^i + \delta, \ldots, x^j, \ldots)$

- Characterizes poverty measures looking like

$$\frac{1}{n_X} \sum_{i=1}^{n_X} f\Big( \sum_{k \in C} g_k(x^i_k) + \sum_{k \in O} v_k(x^i_k) \Big)$$

with $v_k$ as before and "appropriate conditions" on $f$ and $g_k$

## Conclusion

- Characterization of a class of poverty measures featuring:
  - Priority to the worse off poor
  - Both cardinal and ordinal attributes
  - Flexibility in the choice of identification criterion

- Work in progress:
  - Investigate weaker forms of priority
  - Derive practical conditions to check unanimity judgments
  - An application of the latter using EU-SILC data