The accompanying is an illustrative guide to techniques suggested in Alkire and Foster (2007)[1] *and includes sections on identification, measures and weights.*

Introduction

Poverty measurement can be broken down conceptually into two distinct steps: first, the identification step, which defines the criteria for distinguishing poor persons from the non-poor, and second, the aggregation step, by which data on poor persons are brought together into an overall indicator of poverty (Sen, 1976). In unidimensional (income) space, the identification step is typically accomplished by setting a cutoff called the *poverty line* and evaluating whether an individual's income is sufficient to achieve this level. The aggregation step is typically accomplished by selecting a particular *poverty index* or *measure*.

This guide suggests a way to accomplish these tasks in multidimensional poverty measurement. It proposes a method of identification. It then presents four alternative multidimensional poverty measures. Finally, it addresses the issue of weighting of the different dimensions in multidimensional poverty measurement.

IA. Identification: The Deprivation Cutoffs

Consider a matrix y of achievements in d dimensions for n persons. In this example, there are 4 persons and 4 dimensions. The vector z gives the deprivation cutoffs in each dimension; a person is deprived in a given dimension if the achievement is less than or equal to the respective cutoff.

$$\begin{array}{c} \textbf{Persons} \\ y = \begin{bmatrix} 13.1 & 15.2 & 12.5 & 6.5 \\ 14 & 7 & 10 & 12.5 \\ 4 & 5 & 1 & 3 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \ \textbf{Dimension} \qquad z = \begin{bmatrix} 13 \\ 12 \\ 3 \\ 1 \end{bmatrix}$$

If the data are cardinal we can replace all non-deprived entries with zero, and all deprived levels of achievement by normalized gaps $(z_j - y_{ij})/z_j$, to create a matrix of normalized gaps, g_1. Whether the data are cardinal or ordinal, we can construct the matrix g_0 by replacing every positive gap with a 1.

$$g_1 = \begin{bmatrix} 0 & 0 & 0.04 & 0.50 \\ 0 & 0.42 & 0.17 & 0 \\ 0 & 0 & 0.67 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad g_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

[1] Available on www.ophi.org.uk under publications, as Working Paper No. 7

IB. Identification: The Dimensional Cutoff

Using g_0 above, we count vertically down each column to construct a vector c giving the number of deprivations each person experiences. In the above example, c = (0,2,4,1). Next, we fix a dimensional cutoff k, or the minimum number of dimensions in which a person must be deprived in order to be identified as multidimensionally poor. For purposes of illustration, set k = 2; that is, a poor person must be deprived in at least two dimensions. In the above case, 2 out of 4 persons would be identified as poor. Finally, we censor the data of the non-poor, or replace their entries with zeros, to obtain the censored matrix g_0^* and a censored vector c^*.

$$g^0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longrightarrow g_0^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$c = (0 \quad 2 \quad 4 \quad 1) \qquad c^* = (0 \quad 2 \quad 4 \quad 0)$$

IIA. Measure: The Headcount Ratio

Having fixed the method of identification, we now define several multidimensional poverty measures. The **headcount ratio** H is the percentage of persons who are multidimensionally poor; in other words, H = q/n where q is the number of poor. In our example H = 2/4. However, consider what happens if person 2 becomes deprived in another dimension (in bold type below).

$$g_0^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longrightarrow \bar{g}_0^* = \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$c^* = (0 \quad 2 \quad 4 \quad 0) \qquad \bar{c}^* = (0 \quad \mathbf{3} \quad 4 \quad 0)$$

The headcount ratio H remains 2/4. This violates what we call dimensional monotonicity. Intuitively speaking, if person i becomes deprived in a greater number of domains, then overall poverty should increase.

IIB. Measure: The Adjusted Headcount Ratio

What is needed is information on how many deprivations are experienced by the poor. Notice that in our example person 2 is deprived in 2 out of 4 dimensions, while person 3 is deprived in 4 out of 4. Let $A = \Sigma_i (c_i^*/d)/q$ be the average deprivation share of the poor. In our example A = ((2/4)+(4/4))/2 = 6/8. The (dimension) **adjusted headcount ratio** M_0 can be defined as $M_0 = HA$, or the headcount ratio times the average deprivation share. In our example we obtain $M_0 = (1/2)(6/8) = 6/16 = 0.375$. The measure can also be defined as $M_0 = \mu(g_0^*)$, or the mean entry of the matrix g_0^*; in words, M_0 can be viewed as the total number of deprivations experienced by poor persons divided by the highest possible number of deprivations (or dn). In the example there are 6 deprivations and 16 total possible deprivations, hence $M_0 = 6/16$, as before.

If person 2 has an additional deprivation (as in $\bar{g}_0^*$ above), the adjusted headcount will rise to 7/16, so the adjusted headcount ratio respects dimensional monotonicity. This measure also satisfies symmetry, scale invariance, normalization, replication invariance, focus, weak monotonicity, and subgroup decomposability. It can be applied to ordinal data.

IIc. Measure: The Adjusted Poverty Gap

If the data on achievements are cardinal, then one can incorporate information on the depth of deprivations as well. First let us censor the matrix g_1 of normalized gaps to remove the data of the nonpoor; this results in the matrix g_1^*. For our example, this removes person 4's data as illustrated below:

$$g_1 = \begin{bmatrix} 0 & 0 & 0.04 & 0.50 \\ 0 & 0.42 & 0.17 & 0 \\ 0 & 0 & 0.67 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longrightarrow g_1^* = \begin{bmatrix} 0 & 0 & 0.04 & 0 \\ 0 & 0.42 & 0.17 & 0 \\ 0 & 0 & 0.67 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Let G be the average normalized gap in g_1^* across all cases of deprivation. In our example, G = (0.04+0.42+0.17+0.67+1+1)/6 = 0.55. The **adjusted poverty gap** M_1 is defined as M_1 = HAG, or the adjusted headcount ratio multiplied by the average normalized gap. In our example M_1 has the value (0.375)(0.55) or about 0.206. Note that M_1 can also be defined as $M_1 = \mu(g_1^*)$, or the sum of the normalized gaps of the poor divided by the highest possible sum of normalized gaps (or dn). In the example the sum of the deprivations is 3.3, while the maximum possible sum is 16, hence M_1 = 3.3/16, which is about 0.206, as we found above.

Note that if the poverty of any poor persons deepens in any dimension, the adjusted poverty gap will rise, so the adjusted poverty gap respects monotonicity. However, it can also be seen that an increase in deprivation has the same impact no matter the size of the initial deprivation. For example, if in dimension 1 person 3 moves from a normalized gap of 0.04 to one of 0.24 (that is, a further decline of 0.2), this has the same impact as if person 2 moved from a normalized gap of 0.42 to 0.62 in dimension 2. One might argue that the impact should be larger in the latter case.

IId. Measure: The Adjusted FGT measure

In order to be more sensitive to changes of this type, one can base the measure on the squared normalized shortfalls or, more generally, on the $\alpha \geq 0$ powers of the normalized shortfalls. Define the matrix g_α by raising the entries in g_1 to the α power, and let g_α^* be the associated censored matrix. In our example, we would obtain g_2^* and g_α^* as follows:

$$g_2^* = \begin{bmatrix} 0 & 0 & (0.04)^2 & 0 \\ 0 & (0.42)^2 & (0.17)^2 & 0 \\ 0 & 0 & (0.67)^2 & 0 \\ 0 & (1)^2 & (1)^2 & 0 \end{bmatrix} \qquad g_\alpha^* = \begin{bmatrix} 0^\alpha & 0^\alpha & 0.04^\alpha & 0^\alpha \\ 0^\alpha & 0.42^\alpha & 0.17^\alpha & 0^\alpha \\ 0^\alpha & 0^\alpha & 0.67^\alpha & 0^\alpha \\ 0^\alpha & 1^\alpha & 1^\alpha & 0^\alpha \end{bmatrix}$$

The **adjusted FGT** measure M_α is defined as $M_\alpha = \mu(g_\alpha^{*})$, or the sum of the α powers of the normalized gaps of the poor divided by the highest possible sum (which is simply dn). It is clear that for α = 2, the measure M_α deemphasizes the smaller normalized gaps.

The adjusted FGT measure satisfies numerous properties including decomposability, monotonicity (for α > 0), transfer (for α > 1), and dimension monotonicity.

III. Weights:

The above approach implicitly regards every dimension as equally important, a natural assumption when there is no compelling reason to emphasize one dimension over another. However, in certain cases one can argue in favor of different weights for different dimensions. A particularly natural example of this is a nested weighting structure, which partitions the dimensions into equal weighted classes, then applies equal weighting within the members of the class. For example, one might place half of the weight on income and half on the class containing the the non-income capabilities, split equally among the three. Let w be the n by 1 row vector of positive weights that sum to *d*. The weights for the equal weighting structure would be $w^e = (1,1,1,1)$; a nested weighting structure would be $w^n = (2, 2/3, 2/3, 2/3)$.

When dimensions are weighted differentially, the identification step may be reformulated as follows. Given w and g_0, define the associated deprivation share vector by $s_0 = wg_0$. The i[th] coordinate of s_0 is the sum of the weights of the dimensions in which person i is deprived. The dimensional cutoff is, as before, represented by a value *k* between 0 and *d*. Person i is identified as multidimensionally poor if $s_{0i} \geq k$ and nonpoor if $s_{0i} < k$. Denote the respective censored matrix and vector by g_0^* and s_0^*. The matrix g_0 from the above example and the deprivation share vector s_0 obtained from the nested weighting structure w^n are given below. With a cutoff of $k = 2$, person 3 and 4 are now poor and the resulting censored matrix g_0^* and censored deprivation vector $s_0^* = w^n g_0^*$ are also given below.

$$w = \begin{bmatrix} 2 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} \qquad g_0 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \Longrightarrow g_0^* = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$s_0 = (0,\ 4/3,\ 4,\ 2) \qquad s_0^* = (0,\ 0,\ 4,\ 2)$$

The headcount ratio $H = q/n$ is the percentage of the population that is poor; hence $H = ½$ in the above example. The average deprivation share of the poor is now given by $A = \Sigma_i s_{0i}^*/q$ and the adjusted headcount ratio is $M_0 = HA$, or the headcount ratio times the average deprivation share. In our example we have $A = ((4/4)+(2/4))/2 = ¾$ and hence $M_0 = (1/2)(3/4) = 3/8 = 0.375$. The measure can also be defined as a weighted mean of the entries of the matrix g_0^*, namely, $M_0 = \mu(g_0^*;w) = wg_0^*u$ where u is a column vector, each of whose n entries is 1/n. Note that since $s_0^* = wg_0^*$, we have $M_0 = \mu(s_0^*) = \Sigma_i s_{0i}^*/n$. In the example, $\mu(g_0^*;w^n) = \mu(s_0^*) = (1 + ½)/4 = 3/8$ as before. Note that M_0 can be applied to ordinal data.

When variables are cardinal and the weights are general, we can derive M_α for general weights as follows. Let g_α^* be the censored matrix be obtained from g_α using the set of poor identified in s_0 and k. Define $s_\alpha^* = wg_\alpha^*$. Then the adjusted FGT measure is $M_\alpha = \mu(g_\alpha^*;w) = wg_\alpha^*u$, or equivalently, $M_\alpha = \mu(s_\alpha^*) = \Sigma_i s_{\alpha i}^*/n$. For example, when $\alpha = 1$, we obtain the matrix g_1^* of normalized gaps of the poor, and the vector s_1^* containing their (weighted) average depth of deprivations. Or alternatively, when $\alpha = 2$, we obtain the matrix g_2^* of squared normalized gaps of the poor, and the vector s_2^* containing their (weighted) average squared depth of deprivations.

$$w = \begin{bmatrix} 2 \\ 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad g_1^* = \begin{bmatrix} 0 & 0 & 0.04 & 0.50 \\ 0 & 0 & 0.17 & 0 \\ 0 & 0 & 0.67 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad g_2^* = \begin{bmatrix} 0 & 0 & (0.04)^2 & (0.50)^2 \\ 0 & 0 & (0.17)^2 & 0 \\ 0 & 0 & (0.67)^2 & 0 \\ 0 & 0 & (1)^2 & 0 \end{bmatrix}$$

$$s_1^* = (0, \ 0, \ 1.31, \ 1.0)$$

Then $M_1 = \mu(s_1^*) = \Sigma_i s_{1i}^*/4 = 0.32$ and $M_2 = \mu(s_2^*) = \Sigma_i s_{2i}^*/4 = 0.025$.

On the other hand, if the equal weighting w^e structure is used we are back to the original definitions of the adjusted FGT measures. In this case, the deprivation share vector is $s_0 = (0, 2, 4, 1)$ so that with a dimensional cutoff of $k = 2$, person 2 and 3 are multidimensionally poor. The associated censored vectors for $\alpha = 0$ and $\alpha = 1$ are $s_0^* = (0, 2, 4, 0)$ and $s_1^* = (0, 0.1.42, 1.88, 0)$, and so the adjusted headcount ratio is $M_0 = \mu(s_0^*) = (2 + 4)/16 = 6/16 = 0.375$ and the adjusted poverty gap is $M_1 = \mu(s_1^*) = (1.42 + 1.88)/16 = 0.206$. These are the same values that were obtained for M_0 and M_1 when the measures were first introduced above.